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On the Solution of the Time-Dependent Inertial-Frame Equation of Radiative Transfer in Moving Media to $O(v/c)$*

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A stable and efficient mixed-frame method has been formulated for the solution of the time-dependent equation of radiative transfer with full retention of all velocity dependent terms to $O(v/c)$. The method retains the simplicity of the differential operator found in the inertial frame while transforming the absorption and emission coefficients to the comoving frame keeping them isotropic. The method is ideally suited to continuum calculations. To correctly treat the time dependence of the radiation field over fluid-flow time increments, the velocity-dependent terms on the right-hand side of both the transfer and moment equations must be retained for consistency.

Both explicit and two- and three-level implicit schemes have been explored for a variety of time-dependent problems and it has been concluded that an implicit-backward Euler scheme works best for propagating a radiation front, but that these schemes are essentially first-order accurate in the space derivative. A second order scheme was formulated with the method of lines which should provide higher spatial accuracy. The formulation naturally couples to hydrodynamics in both the Eulerian and Lagrangian formulations for application to astrophysical flows.

It is shown that for uniform flow between the fixed and comoving frames, the solution of the Lorentz transformation of the integrated moments provides a powerful check on the formulation and solution.

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I. Introduction

In moving media, the equation of radiative transfer contains velocity-dependent terms, the largest of which are $O(v/c)$, that account for aberration, advection, and Doppler shifts. In most existing solutions of transfer problems these terms are ignored, even though they clearly can be important in high-velocity flows. As we shall show below, the $O(v/c)$ terms should be retained: (a) when characteristic time scales in the flow are so short that the transfer equation itself (as opposed to moment equations) must be treated as being explicitly time dependent, and (b) in order to obtain strict consistency among various forms of the energy equation for the combined matter-radiation fluid. We therefore wish to develop stable and efficient techniques that both account for the time dependence of the flow, and include all velocity-dependent terms to $O(v/c)$.

In formulating the transfer problem to $O(v/c)$ one has two basic choices of reference frame: (1) the comoving-fluid frame (Lagrangian frame), or (2) an inertial laboratory frame. Both of these choices have advantages and disadvantages.

A. *The Comoving Frame*

Consider first the comoving frame. Here one is working in a *noninertial* spacetime, because different material elements (to which the coordinate markers are attached) experience accelerations with respect to one another. In this event one must introduce additional terms into the transfer equation to account for differential motions of the fluid over the path of a photon. Derivations of the appropriate comoving-frame equations have been given by Castor [3], using Lindquist's [10] formalism, by Buchler [2], using invariance of the photon-Boltzmann equation, and by Mihalas [13]; the former two carry only $O(v/c)$ terms and work on a true Lagrangian frame, the latter carries terms to all orders in (v/c), but works in an inertial spacetime that can be reduced to a Lagrangian frame only when terms of higher order than (v/c) are omitted.

The basic advantages of the comoving frame is that it is the *proper frame* of the radiating fluid in the relativistic sense. Thus it is the natural frame for specifying the thermodynamic properties of the fluid. Moreover, it is the frame in which atomic absorption and emission coefficients are isotropic, and in which partial-redistribution functions (e.g., for Compton scattering) are most easily written. The disadvantage of this approach is that it leads to complicated equations, containing derivatives with respect to time, space coordinates, angle coordinates, and frequency, which are hard to solve. Therefore, in practice, on a few solutions have been obtained, all for very simple one-dimensional flows; it is not at all clear at the present time whether, and how, this method could be applied in two- or three-dimensional flows.

The comoving-frame equations are easily "solved" in the *diffusion limit* where the photon mean-free path $\lambda_p \ll l$, a characteristic length in the flow. Here one knows that the radiation field thermalizes and becomes isotropic so that one can discard the transfer equation and can use equilibrium values for the radiation-energy density E_R

and the radiation pressure p_R, and can write the radiation flux as $F_R = K_R \nabla T$. One can then treat the fluid as having a total specific-energy density $e = e_g + (E_R/\rho)$, total pressure $p = p_g + p_R$, and a "conductive" energy flux with conductivity K_R.

When transport effects are important, however, a direct solution of the transfer equation is required. If one demands complete consistency, including all velocity-dependent terms, the solution is very difficult and must be carried out along *curved* rays representing the photon paths in the noninertial spacetime; in practice this has been done, thus far, only for one-dimensional steady expansion of a spherical medium (Mihalas *et al.* [16], Mihalas [13]). For the case of line transfer the problem can be somewhat simplified because one can show that the frequency-derivative term is effectively amplified to $O(v/v_{\text{therm}})$, where v_{therm} is a characteristic thermal velocity in the line; here one can omit all $O(v/c)$ terms in comparison with the $(\partial/\partial \nu)$ term, and a practical solution can be developed (Mihalas *et al.* [15]), at least for an expanding medium. The comoving frame is especially advantageous for this problem because one can choose, once and for all, a small set of comoving-frame frequencies to sample the line profile regardless of how large the flow velocities become. For continuum transfer the equations have been solved very approximately, on the assumption that the $(\partial/\partial \nu)$ term can be dropped; in practice, one then solves only the moment equations and uses a static, velocity-independent solution of the transfer equation to update variable Eddington factors (Falk and Arnett [5]). A more rigorous treatment is possible (Mihalas [13]) but is too costly for inclusion into a hydrodynamics calculation.

In summary, the comoving-frame formulation is physically elegant but computationally complex, and only limited progress has been made with it. We therefore examine other formulations.

B. *The Inertial Frame*

In the inertial frame, spacetime is flat, and the differential operator in the transfer equation remains relatively simple, containing only time and space derivatives. As a result, it becomes possible to handle more complicated velocity fields in the flow, and possibly to treat multidimensional flows, at least in an Eulerian framework. The fundamental disadvantage of the inertial frame is that material properties, specifically the absorption and emission coefficients become *anisotropic* because a photon with lab-frame frequency traveling in direction $\mathbf{n}$ has a comoving-frame frequency·

$$\nu_0 = \nu\gamma(1 - \mathbf{n} \cdot \mathbf{v}/c), \tag{1.1}$$

and travels in direction

$$\mathbf{n}_0 = (\nu/\nu_0)\{\mathbf{n} - (\mathbf{v}/c)[\gamma - (\gamma^2 \mathbf{n} \cdot \mathbf{v}/(\gamma + 1)c)]\}. \tag{1.2}$$

This anisotropy is particularly troublesome for spectral lines where a small frequency-shift induces a large change in opacity. The total bandwith that must be treated with good frequency resolution in, say, a line-formation problem can become very large, the intrinsic linewidth $\pm(\nu_0 v_{\max}/c)$, where $v_{\max}$ is the maximum flow speed; this can be quite costly. Worse, scattering integrals become two dimensional (angles and frequencies) even under the simplifying assumption of complete redistribution. Despite its disadvantages, this approach has been used to treat (time-independent) line-transfer problems.

An interesting alternative to the approaches described above is a *mixed-frame* scheme in which one uses an inertial spacetime and expresses radiation quantities, angles, and frequencies in the lab frame, but expresses opacities and emissivities in the comoving frame via an $O(v/c)$ expansion. This approach is discussed by Fraser [6], Pomraning [17], and Hsieh and Spiegel [8]. This approach retains the advantage of simplicity of the differential operator so that complex flows (e.g., shocks) and multidimensional geometries can, in principle, be handled, and gains the additional advantage that the absorption and emission coefficients are now effectively isotropic. The disadvantages of this approach are that it is difficult to treat scattering with partial redistribution, and the expansion procedure fails in spectral lines because of the rapid-frequency variation of the emissivity and absorptivity within the line profile. In addition, the viewpoint is fundamentally Eulerian, which is less convenient, at least for one-dimensional flows, than a Lagrangian formulation (though, as we shall see, it is possible to recast the equations into Lagrangian form).

Despite the fact that the mixed-frame scheme was one of the first to be formulated, there seems to have been little work devoted to actually solving the equations, except in the diffusion limit (Hsieh and Spiegel [8]), and the approach remains essentially unexplored. Our goal in this report is to demonstrate that practical schemes can be developed for treating he hybrid time-dependent transfer equation including $O(v/c)$ terms both stably and efficiently, and to sketch how they can be coupled into radiation-hydrodynamics calculations; computations using these methods for actual flows remains work for the future.

In § II we shall develop the mixed-frame equation of transfer and the frequency-dependent and frequency-integrated moment equations. In § III we shall consider the relative sizes of the terms in the transfer equation and the requirement to keep all v/c terms to obtain exact consistency with the energy equation for a radiating fluid. In § IV we shall discuss our computational approach in Feautrier variables and we include a 2nd-order formulation. We shall present several schemes for the method of solution in § V and provide test calculations. In § VI we shall indicate how the transfer equation to $O(v/c)$ can be coupled to the hydrodynamic equations in both a Eulerian and a Lagrangian formulation.

II. The Transfer Equation and Its Moments

A. *The Transfer Equation*

The transfer equation for radiation of frequency ν traveling in direction $\mathbf{n}$ in a moving medium is

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + n^j \frac{\partial}{\partial x^j}\right) I(\mathbf{n}, \nu) = \eta(\mathbf{n}, \nu) - \kappa(\mathbf{n}, \nu)\, I(\mathbf{n}, \nu), \tag{2.1}$$

where η is the thermal emissivity and κ the true absorptivity of the material (i.e., scattering has been omitted). Our approach is to leave both the differential operator and the specific intensity in the inertial frame but to express the material properties on the right-hand side in the fluid frame.

By demanding Lorentz invariance of the transfer equation Thomas [19] showed that the general transformation relations for κ and ν between the inertial and comoving frames are

$$\kappa(\mathbf{n}, \nu) = (\nu_0/\nu)\, \kappa_0(\nu_0) \tag{2.2}$$

and

$$n(\mathbf{n}, \nu) = (\nu/\nu_0)^2 \eta_0(\nu_0), \tag{2.3}$$

where quantities with subscript "0" are measured in the fluid frame. Consider only the low-velocity limit, and, henceforth, retain terms only to $O(v/c)$. Then

$$(\nu/\nu_0) = 1 + \mathbf{n} \cdot \mathbf{v}/c, \tag{2.4}$$

$$\kappa(\mathbf{n}, \nu) = \kappa_0(\nu) - (\mathbf{n} \cdot \mathbf{v}/c)[\kappa_0(\nu) + \nu(\partial \kappa_0/\partial \nu)], \tag{2.5}$$

and

$$\eta(\mathbf{n}, \nu) = \eta_0(\nu) + (\mathbf{n} \cdot \mathbf{v}/c)[2\eta_0(\nu) - \nu(\partial \eta_0/\partial \nu)]. \tag{2.6}$$

Using (2.5) and (2.6) in (2.1) we obtain the mixed-frame transfer equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + n^j \frac{\partial}{\partial x^j}\right) I(\mathbf{n}, \nu) = \eta_0(\nu) - \kappa_0(\nu)\, I(\mathbf{n}, \nu) + \left(\frac{n_j v^j}{c}\right)\left\{2\eta_0(\nu) - \nu\frac{\partial \eta_0}{\partial \nu} + \left[\kappa_0(\nu) + \nu\frac{\partial \kappa_0}{\partial \nu}\right] I(\mathbf{n}, \nu)\right\}. \tag{2.7}$$

B. Scattering Terms

Scattering terms in the mixed-frame formulation are complicated; a detailed discussion is given by Fraser [6]. Here we shall consider only Thomson scattering, which we take to be grey, coherent, and isotropic in the comoving frame. The sink term to be added to the right-hand side of (2.7) is simply

$$-\sigma(\mathbf{n}, \nu)\, I(\mathbf{n}, \nu) = -(\nu_0/\nu)\, \sigma_0 I(\mathbf{n}, \nu) = -\sigma_0(1 - n_j v^j/c)\, I(\mathbf{n}, \nu). \tag{2.8}$$

In the comoving frame the source term is

$$\eta_0^S(\nu_0) = (\sigma_0/4\pi) \oint I_0(\mathbf{n}_0, \nu_0)\, d\omega_0, \tag{2.9}$$

hence, in the observer's frame

$$\eta^S(\mathbf{n}, \nu) = (\nu/\nu_0)^2 \eta_0^S(\nu_0) = (\nu/\nu_0)^2 (\sigma_0/4\pi) \oint I_0(\mathbf{n}_0, \nu_0)\, d\omega_0, \tag{2.10}$$

where we now regard $\mathbf{n}$ and ν as fixed, and ν_0 and $\mathbf{n}_0$ on the right-hand side to be given by (1.1) and (1.2). That is,

$$\nu_0 = \nu(1 - \mathbf{n} \cdot \mathbf{v}/c). \tag{2.11}$$

We now wish to re-express the integral in terms of the inertial-frame radiation field and inertial-frame angles. Thomas]19] showed that, in general,

$$I_0(\mathbf{n}_0, \nu_0) = (\nu_0/\nu')^3 I(\mathbf{n}', \nu') \tag{2.12}$$

and

$$d\omega_0 = (\nu'/\nu_0)^2\, d\omega'. \tag{2.13}$$

Here

$$\nu' = \nu_0(1 + \mathbf{n}' \cdot \mathbf{v}/c) = \nu[1 + (\mathbf{n}' \cdot \mathbf{v} - \mathbf{n} \cdot \mathbf{v})/c], \tag{2.14}$$

where the second equality follows from (2.11). Thus $O(v/c)$,

$$\begin{aligned} \oint I_0(\mathbf{n}_0, \nu_0)\, d\omega_0 &= \oint \left(\frac{\nu_0}{\nu'}\right) \left[I(\nu) + \frac{\nu}{c}(\mathbf{n}' \cdot v - \mathbf{n} \cdot \mathbf{v}) \frac{\partial I}{\partial \nu}\right] d\omega' \\ &= \oint \left[I(\nu)\left(1 - \frac{\mathbf{n}' \cdot \mathbf{v}}{c}\right) + \frac{\nu}{c}(\mathbf{n}' \cdot \mathbf{v} - \mathbf{n} \cdot \mathbf{v}) \frac{\partial I}{\partial \nu}\right] d\omega' \\ &= 4\pi \left[J(\nu) - \frac{v_j H^j(\nu)}{c} + \left(\frac{\nu v_j}{c}\right)\left(\frac{\partial H^j}{\partial \nu} - n^i \frac{\partial J}{\partial \nu}\right)\right]. \end{aligned} \tag{2.15}$$

Here we have introduced the mean intensity

$$J(\nu) \equiv (1/4\pi) \oint I(\mathbf{n}, \nu)\, d\omega \tag{2.16}$$

and the Eddington flux

$$H^i(\nu) \equiv (1/4\pi) \oint I(\mathbf{n}, \nu)\, n^i\, d\omega. \tag{(2.17)}$$

Thus from (2.8), (2.10), (2.11), and (2.15) we find that to $O(v/c)$ the transfer equation including Thomson scattering becomes

$$\frac{1}{c}\frac{\partial}{\partial t}+\eta^j\frac{\partial}{\partial x^j}I(\mathbf{n},\nu)=\eta_0(\nu)+\sigma_0 J(\nu)-[\kappa_0(\nu)+\sigma_0]\,I(\mathbf{n},\nu)$$
$$+\left(\frac{n_j v^j}{c}\right)\left\{2\eta_0(\nu)-\nu\frac{\partial\eta_0}{\partial\nu}+\sigma_0\left[2J(\nu)-\nu\frac{\partial J}{\partial\nu}\right]\right.$$
$$\left.+\left[\kappa_0(\nu)+\nu\frac{\partial\kappa_0}{\partial\nu}+\sigma_0\right]I(_\mathrm{n},\nu)\right\}-\left(\frac{\sigma_0 v_j}{c}\right)\left[H^j(\nu)-\nu\frac{\partial H^j}{\partial\nu}\right]. \qquad (2.18)$$

It is apparent that the scattering term significantly complicates the problem because it introduces moments of the radiation field and their frequency derivatives into the problem, so that the transfer equation is an integro-partial differential equation.

C. *Frequency-Dependent Moment Equations*

We obtain frequency-dependent moment equations by integrating (2.18) over solid angle against powers of $\mathbf{n}$. Thus integrating over $d\omega$ we obtain the zeroth-moment equation

$$\frac{1}{c}\frac{\partial J(\nu)}{\partial t}+\frac{\partial H^i(\nu)}{\partial x^i}=\eta_0(\nu)-\kappa_0(\nu)\,J(\nu)+\left[\kappa_0(\nu)+\nu\frac{\partial\kappa_0}{\partial\nu}\right]\frac{v_i H^i(\nu)}{c}$$
$$+\left(\frac{\sigma_0\nu}{c}\right)v_i\frac{\partial H^i}{\partial\nu}. \qquad (2.19)$$

Similarly, integrating against $n^i\,d\omega$ and noting that

$$\oint n^i n^j\,d\omega=\frac{4\pi}{3}\delta^{ij} \qquad (2.20)$$

we obtain the first moment equation

$$\frac{1}{c}\frac{\partial H^i(\nu)}{\partial t}+\frac{\partial K^{ij}(\nu)}{\partial x^j}=-[\kappa_0(\nu)+\sigma_0]\,H^i(\nu)$$
$$+\frac{v^i}{3c}\left\{2\eta_0(\nu)-\nu\frac{\partial\eta_0}{\partial\nu}+\sigma_0\left[2J(\nu)-\nu\frac{\partial J}{\partial\nu}\right]\right\}$$
$$+\frac{1}{c}\left[\kappa_0(\nu)+\nu\frac{\partial\kappa_0}{\partial\nu}+\sigma_0\right]v_j K^{ij}(\nu), \qquad (2.21)$$

where the tensor K^{ij} is defined as

$$K^{ij}(\nu)\equiv(1/4\pi)\oint I(n,\nu)\,n^i n^j\,d\omega. \qquad (2.22)$$

D. *Frequency-Integrated Moment Equations*

Next, by integrating over frequency we obtain frequency-integrated moment equations, which are the expressions that appear in the equations of radiation hydrodynamics. From (2.19) we obtain after integration by parts the radiation-energy equation

$$\frac{\partial E_{\mathrm{R}}}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi \int_0^\infty [\eta_0(\nu) - \kappa_0(\nu) J(\nu)]\, d\nu + \frac{v_i}{c} \int_0^\infty \left[\kappa_0(\nu) + \nu \frac{\partial \kappa_0}{\partial \nu} - \sigma_0\right] F^i(\nu)\, d\nu. \tag{2.23}$$

Similarly from (2.21) we obtain the radiation-momentum equation

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P_{\mathrm{R}}^{ij}}{\partial x^j} = -\frac{1}{c}\int_0^\infty [\kappa_0(\nu) + \sigma_0]\, F^i(\nu)\, d\nu + \frac{v^i}{c}\left[\frac{4\pi}{c}\int_0^\infty \eta_0(\nu)\, d\nu + \sigma_0 E_{\mathrm{R}}\right] + \frac{v_j}{c}\left\{\int_0^\infty \left[\kappa_0(\nu) + \nu\frac{\partial \kappa_0}{\partial \nu}\right] P_{\mathrm{R}}^{ij}(\nu)\, d\nu + \sigma_0 P_{\mathrm{R}}^{ij}\right\}. \tag{2.24}$$

Here we use the radiation-energy density

$$E_{\mathrm{R}} \equiv \frac{4\pi}{c} J = \frac{4\pi}{c}\int_0^\infty J(\nu)\, d\nu, \tag{2.25}$$

the radiative flux

$$\mathbf{F} \equiv 4\pi \mathbf{H} = 4\pi \int_0^\infty H(\nu)\, d\nu, \tag{2.26}$$

and the radiation-pressure tensor

$$P_{\mathrm{R}}^{ij} \equiv (4\pi/c)\, K^{ij} = (4\pi/c)\int_0^\infty K^{ij}(\nu)\, d\nu. \tag{2.27}$$

Although these equations specify the dynamics of the radiation field in principle, in practice they suffer from the usual closure problem. Approximate closure schemes in the diffusion limit have been developed by Thomas [19], Masaki [11], and Hsieh and Spiegel [8]. But in the optically thin limit, only by a solution of the full angle-frequency dependent Eq. (2.18) can one close the system (2.23) and (2.24), say in terms of variable Eddington factors.

E. *Grey Material*

We can significantly simplify the problem for grey material, in which $\kappa_0(\nu) \equiv \kappa_0$, a constant. We further assume LTE so that $\eta_0(\nu) = \kappa_0 B_\nu(T_0)$, where T_0 is the material temperature in the comoving frame and B_ν is the Planck function. Then the grey-transfer equation is obtained by integrating (2.18) over frequency, whence we have

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \eta^j \frac{\partial}{\partial x^j}\right) I(\mathbf{n}) = \kappa_0 B + \sigma_0 J - (\kappa_0 + \sigma_0) I(\mathbf{n}) + (n_j v^j/c)[(\kappa_0 + \sigma_0) I(\mathbf{n}) + 3(\kappa_0 B + \sigma_0 J)] - (2\sigma_0/c) v_j H^j. \tag{2.28}$$

Taking the zeroth-angular moment of (2.28), we find the radiation-energy equation for grey material

$$\frac{\partial E_R}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi\kappa_0 B - \kappa_0 c E_R + \frac{(\kappa_0 - \sigma_0)}{c} v_i F^i, \tag{2.29}$$

and by taking the first moment, we find the radiative-momentum equation

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P_R^{ij}}{\partial x^j} = -\frac{1}{c}(\kappa_0 + \sigma_0) F^i + \left(\frac{4\pi\kappa_0 B}{c} + \sigma_0 E_R\right)\frac{v^i}{c} + \frac{\kappa_0 + \sigma_0}{c} v_j P_R^{ij}. \tag{2.30}$$

Alternatively, these grey-moment equations can be derived from Eqs. (2.23), (2.24) by inspection. To $O(v/v)$, Eq. (2.29) agrees with Eq. (4.11) of Hsieh and Spiegel [8], and (2.30) with their Eq. (4.12).

III. Importance of (v/c) Terms

Let us now investigate the circumstances under which the (v/c) terms in the equations should be retained. The basic conclusion is that: (1) they are required for consistency if the time dependence of the radiation field or its moments is at all important, and (2) various forms of the energy equation are consistent only if the (v/c) terms are retained.

A. *Relative Sizes of Terms in the Transfer Equation*

First consider the order-of-magnitude relative sizes of terms in Eqs. (2.28)–(2.30), ignoring all scattering terms. In making estimates we shall use the fact that in the free-flow limit $J \approx H \approx K$, hence $E_R \approx p_R$ and $F \approx cE_R$. In the diffusion limit

$J = 3K = B$, and $H \ll J$; in this limit the Lorentz transformation for the flux [cf. Eq. (3.2)] shows that $F \sim vE_R + F_0$, where F_0 is the flux evaluated in the comoving frame.

Starting with the transfer equation (2.28), we see that for time-increments appropriate to fluid flow, i.e., $\Delta t \sim \Delta x/v$, the $\partial/\partial t$ on the left-hand side is obviously $O(v/c)$ relative to the spatial derivative terms. Therefore, if time dependence is important, the solution should be carried out to $O(v/c)$. On the right-hand side, in the free-flow limit all terms beyond the usual absorption and emission terms are $O(v/c)$ relative to those terms, and thus should be retained for consistency with the $(\partial/\partial t)$ term. In the diffusion limit $I \rightarrow B[1 + O(\lambda/\Delta x)^2]$, where λ is the photon mean-free-path and Δx is a characteristic flow length. Because $\lambda \ll \Delta x$, the absorption and emission terms cancel almost identically, and become $O(\lambda/\Delta x)$ relative to the spatial derivative. In contrast, the v-dependent terms are of order $(v/c)(\Delta x/\lambda)$ relative to the spatial derivative, which shows that: (1) they are much more important than the $(\partial/\partial t)$ term, and (2) that they can dominate the absorption-emission terms if $(v/c)(\Delta x/\lambda)^2 \geqslant 1$. We therefore conclude that in the diffusion limit it is essential to retain the v/c terms on the right-hand side.

For the zeroth-moment equation, in the free-flow limit we see that on the left-hand side the term $\partial E_R/\partial t$ is $O(v/c)$ relative to $\nabla \cdot \mathbf{F}$. On the right-hand side the usual first two terms are of order $c\kappa E_R$ while the last is of order $v\kappa E_R$ and hence of $O(v/c)$ relative to the other two. Therefore, a consistent time-dependent solution is obtained only if all terms are retained. In the diffusion limit both terms on the left-hand side are of the same order, namely, $vE_R/\Delta x$. On the right-hand side, the difference between the first two terms is of order $c\lambda E_R/\Delta x^2$ and the last is of order $v^2 E_R/c\lambda$. Thus, relative to the spatial and time derivatives the absorption-emission terms are $O[(c/v)(\lambda/\Delta x)]$ while the last term is $O[(v/c)(\Delta x/\lambda)]$; clearly the v-dependent term becomes important for opaque material ($\lambda \ll 1$) and moderate flow-speeds, i.e., when $(v/c)(\Delta x/\lambda) \sim 1$ or greater.

For the first-order moment equation, in the diffusion limit all three terms on the right-hand side are of the same order, namely, $(v\kappa E_R/c) = (vE/c\lambda)$, and hence, all must be retained. On the left-hand side the term $c^{-2}(\partial F/\partial t)$ is only $O(v^2/c^2)$ relative to $(\partial P_R/\partial x)$ and, hence, can be dropped in the diffusion limit. In the free-flow limit, however, the $(\partial F/\partial t)$ term becomes $O(v/c)$ relative to $(\partial P_R/\partial x)$, as do the v-dependent terms on the right-hand side relative to $\kappa_0 F/c$. We therefore, again conclude that consistency is obtained only if all terms are retained in the solution.

In summary, if one wishes to treat the time dependence of the radiation field over time increments appropriate to fluid flow correctly, then the velocity-dependent terms on the right-hand side of the transfer and moment equations must be retained for consistency, for they are of the same order, or larger, as the $\partial/\partial t$ term. If we are interested in a pure radiation-flow problem the appropriate time increments are $\Delta t \sim \Delta x/c$. Here the time-dependent terms are $O(1)$ relative to the spatial derivative. Unless we specifically demand a solution accurate to $O(v/c)$, the velocity-dependent terms on the RHS may be omitted.

B. *Consistency of the Energy Equation*

In the energy and momentum equations (2.29) and (2.30) there are two kinds of $O(v/c)$ terms: (1) the velocity-dependent terms on the right-hand side and the $(\partial/\partial t)$ terms on the left-hand side, and (2) those that discriminate between radiation quantities measured in the comoving and inertial frames. By Lorentz transformation one finds for the latter

$$E_{\mathrm{R}} = E_{\mathrm{R}0} + 2v_i F_0^i/c^2 + O(v^2/c^2), \tag{3.1}$$

$$F^i = F_0^j + E_{\mathrm{R}0} v^i + v_j P_{\mathrm{R}0}^{ij} + O(v^2/c^2), \tag{3.2}$$

and

$$P_{\mathrm{R}}^{ij} = P_{\mathrm{R}0}^{ij} + (v^i F_0^j + v^j F_0^i)/c^2 + O(v^2/c^2). \tag{3.3}$$

These transformations apply only for the frequency-integrated moments. We shall now show that if and only if we retain all these $O(v/c)$ terms can we obtain strict consistency among various forms of the energy equation for the combined radiation-material fluid.

The equation for *overall energy conservation* for the radiating fluid, correct to $O(v/c)$ is

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho v^2 + E_{\mathrm{R}}) + \frac{\partial}{\partial x_i}[(\rho e + \tfrac{1}{2}\rho v^2 + p)\, v^i + F^i] = v_i f^i \tag{3.4}$$

or

$$\rho \frac{D}{Dt}(e + \tfrac{1}{2}v^2) + \frac{\partial}{\partial x^i}(p v^i) = v_i f^i - \frac{\partial E_{\mathrm{R}}}{\partial t} + \frac{\partial F^i}{\partial x^i}, \tag{3.5}$$

where f^i represents external forces on the radiating fluid [20, Eq. (11); 17, Eq. (9.84); 12, Eq. 15-119]. Here e is the specific internal energy of the fluid, p is the fluid pressure, ρ is the fluid density, and f^i is the force per unit volume acting on the fluid.

The *momentum equation* correct to $O(v/c)$ is

$$\rho \frac{Dv^i}{Dt} = f^i - \frac{\partial p}{\partial x^i} - \left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P_{\mathrm{R}}^{ij}}{\partial x^j}\right) + \frac{v^i}{c^2}\frac{\partial E_{\mathrm{R}}}{\partial t} + \frac{\partial F^j}{\partial x^j} \tag{3.6}$$

[20, Eq. (8); Eq. (15-109)]. Note in passing that Pomraning [17, Eq. (9.83)] omits the last term on the right-hand side of Eq. (3.6).

Forming the product of (3.6) with v_i we obtain the *mechanical-energy equation*

$$\rho \frac{D}{Dt}(\tfrac{1}{2}v^2) = v_i f^i - v^i \frac{\partial p}{\partial x^i} - v_i\left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P_{\mathrm{R}}^{ij}}{\partial x^j}\right) + O(v^2/c^2), \tag{3.7}$$

then subtracting (3.7) from (3.5) we obtain the *gas-energy equation*

$$\rho \left[\frac{De}{Dt} + p \frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = v_i \left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P_R^{ij}}{\partial x^j}\right) - \left(\frac{\partial E_R}{\partial t} + \frac{\partial F^i}{\partial x^i}\right), \tag{3.8}$$

where we used the continuity equation $(D\rho/Dt) = -\rho(\partial v^i/\partial x^i)$. Now using (2.29) and (2.30) in (3.8) we obtain

$$\rho \left[\frac{De}{Dt} + p \frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \kappa_0 c E_R - 4\pi\kappa_0 B - \frac{2\kappa_0}{c} v_i F^i, \tag{3.9}$$

then using (3.1) we obtain

$$\rho \left[\frac{De}{Dt} + p \frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \kappa_0 c E_R - 4\pi\kappa_0 B = 4\pi\kappa_0(J_0 - B), \tag{3.10}$$

which is the correct gas-energy equation in the comoving frame [Castor [3, Eq. (43)]. Had either the (v/c) term on the right-hand side of Eq. (2.29) or the (v/c) terms in Eq. (3.1) been ignored, this exact reduction would not have occurred, and we would have been left with an extra term of the form $(\kappa_0 v_i f^i/c)$, which is the rate of work done by radiation forces on the material.

Similarly, starting from the zeroth-moment equation (2.29), we can derive the first law of thermodynamics for the radiation field. Thus using Eqs. (3.1)–(3.3) in Eq. (2.29), and noting that for time intervals $\Delta t \sim \Delta x/c$ the (v/c) terms in (3.1) lead to terms of $O(v^2/c^2)$ in $\partial E_R/\partial t)$, we find

$$\begin{aligned} &\frac{\partial E_{R0}}{\partial t} + \frac{\partial F_0^i}{\partial x^i} + v^i \frac{\partial E_{R0}}{\partial x^i} + E_{R0}\frac{\partial v^i}{\partial x^i} + P_{R0}^{ij}\frac{\partial v_j}{\partial x^i} + v_j \frac{\partial P_{R0}^{ij}}{\partial x^i} \\ &\quad = 4\pi\kappa_0(B - J_0) - (\kappa_0 + \sigma_0)(v_i F^i/c). \end{aligned} \tag{3.11}$$

Grouping terms, invoking continuity, and using Eq. (2.30) to write

$$v_j \frac{\partial P_{R0}^{ij}}{\partial x^i} = -(\kappa_0 + \sigma_0)\frac{v_i F^i}{c} + O(v^2/c^2) \tag{3.12}$$

we find that Eq. (3.11) reduces to

$$\rho \frac{D}{Dt}\left(\frac{E_{R0}}{\rho}\right) + P_{R0}^{ij}\frac{\partial v_j}{\partial x^i} = 4\pi\kappa_0(B - J_0) - \frac{\partial F_0^i}{\partial x^i} \tag{3.13}$$

which is the first law of thermodynamics for the radiation field [of Castor [3, Eq. (35)]]. The first term is the rate of increase of radiant energy per unit mass. The second is the contraction of the radiation-stress tensor with the velocity-gradient tensor and, hence is the rate of work done by radiation pressure; this becomes more evident for an isotropic field in which

$$P^{ij}_{\mathrm{R}0} = P_{\mathrm{R}_0}\delta^{ij}, \qquad \text{whence} \quad P^{ij}_{\mathrm{R}_0}(\partial v_j/\partial x^i) \rightarrow P_{\mathrm{R}_0}(\partial v^i/\partial x^i) = \rho P_{\mathrm{R}_0}[D(1/\rho)/Dt].$$

The terms on the right-hand side are the net rate of delivery of energy by the material to the radiation field minus the net rate of energy flux out of a volume. Again, if we had omitted any (v/c) terms in Eqs. (3.1)–(3.3) or on the right-hand side of Eq. (2.29), the reduction to Eq. (3.13) would not have been exact, and again the error have been proportional to the rate of work done by radiation forces on the material. (See also the discussion in [3, p. 790]).

Finally, the sum of (3.10) and (3.13) give the first law of thermodynamics for the combined radiation-material fluid.

In summary, exact consistency of various forms of the energy equation for a radiating fluid requires retention of all v/c terms affecting the radiation field.

IV. Computational Approach

A. *Preliminaries*

To investigate computational methods for solving the transfer equation including time- and velocity-dependent terms we simplify the problem by assuming: (1) one-dimensional planar geometry, (2) no scattering, and (3) LTE so that $\eta_0(\nu) = \kappa_0(\nu) B_\nu$. For economy of notation we shall henceforth drop the affix "0" on the material properties. The basic transfer equation follows from Eq. (2.18) and is

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mu\frac{\partial I_\nu}{\partial z} = \eta_\nu - \kappa_\nu I_\nu + \mu\beta(\tilde{\kappa}_\nu I_\nu + \tilde{\eta}_\nu), \tag{4.1}$$

where

$$\tilde{\kappa}_\nu \equiv \kappa_\nu + \nu(\partial\kappa_\nu/\partial\nu), \tag{4.2}$$

and

$$\tilde{\eta}_\nu \equiv 2\eta_\nu - \nu(\partial\eta_\nu/\partial\nu). \tag{4.3}$$

Here, $I_\nu = I(z, \mu, \nu)$, $\mu = \cos\theta$, where θ is the angle between the direction of a ray and the positive z axis, and $\beta \equiv (v/c)$, where v is the velocity along the z axis. The corresponding equation for spherical geometry has the same right-hand side, and on the left-hand side $\mu(\partial I_\nu/\partial z)$ is replaced by $\mu(\partial I_\nu/\partial r) + r^{-1}(1-\mu^2)(\partial I_\nu/\partial\mu)$ or, equivalently, $(\mu/r^2)(\partial/\partial r)(r^2 I_\nu) + (1/r)(\partial/\partial\mu)[(1-\mu^2) I_\nu]$; we shall not discuss spherical geometry further in this paper.

B. *Multigroup Equations*

It is trivial to derive both frequency-dependent and frequency-integrated moments (4.1), but before doing so it is worthwhile to simplify to a *multigroup formulation.* Thus, on (ν_k, ν_{k+1}) let the opacity have a *constant* value $\kappa_\nu \equiv \kappa_k$. Then, integrating (4.1) over (ν_k, ν_{k+1}) and assuming LTE we have

$$\frac{1}{c}\frac{\partial I_k}{\partial t} + \mu \frac{\partial I_k}{\partial z} = \kappa_k(B_k - I_k) + \mu\beta(\kappa_k I_k + \tilde{\eta}_k), \tag{4.4}$$

where

$$I_k \equiv \int_{\nu_k}^{\nu_{k+1}} I_\nu \, d\nu \tag{4.5}$$

$$B_k \equiv \int_{\nu_k}^{\nu_{k+1}} B_\nu \, d\nu \tag{4.6}$$

and

$$\tilde{\eta}_k \equiv \int_{\nu_k}^{\nu_{k+1}} \tilde{\eta}_\nu d\nu = 3\int_{\nu_k}^{\nu_{k+1}} \eta_\nu \, d\nu + [\nu_{k+1}\eta(\nu_{k+1}) - \nu_k \eta(\nu_k)]. \tag{4.7}$$

In Eq. (4.7) the term in the brackets vanishes identically when the integration range extends over $(0, \infty)$. To achieve this in the multigroup formulation we have two options: (1) we could write $\eta(\nu_k) \equiv \frac{1}{2}(\kappa_{k-1}B_{k-1} + \kappa_k B_k)$, in which case we have

$$\tilde{\eta}_k = 3\kappa_k B_k + \tfrac{1}{2}[\nu_{k+1}(\kappa_k B_k + \kappa_{k+1}B_{k+1}) - \nu_k(\kappa_{k-1}B_{k-1} + \kappa_k B_k)] \equiv 3\kappa_k \tilde{B}_k, \tag{4.8}$$

(2) we would simply *drop* the term in the brackets, which will be small if $(\nu_{k-1} - \nu_k)$ is not too large. In either case we use the notation of Eq. (4.8) to write, finally,

$$\frac{1}{c}\frac{\partial I_k}{\partial t} + \mu \frac{\partial I_k}{\partial z} = \kappa_k(B_k - I_k) + \mu\beta\kappa_k(I_k + 3\tilde{B}_k), \tag{4.9}$$

which yields the zeroth-moment equation

$$\frac{1}{c}\frac{\partial J_k}{dt} + \frac{\partial H_k}{\partial z} = \kappa_k(B_k - J_k) + \beta\kappa_k H_k \tag{4.10}$$

and the first-order moment equation

$$\frac{1}{c}\frac{\partial H_k}{\partial t} + \frac{\partial K_k}{\partial z} = -\kappa_k H_k + \beta\kappa_k(K_k + \tilde{B}_k). \tag{4.11}$$

The sum of (4.10) and (4.11) over all k gives integrated-moment equations. In the limit of a single group, κ = constant on $\nu = (0, \infty)$ and we recover the grey equations.

C. *Transformation to Feautrier Variables*

Equations (4.10) and (4.11) could be taken as the basic equations to be solved. But, of course, they suffer from the closure problem because two equations contain three moments. To achieve closure we could introduce the variable Eddington factor f_k defined such that

$$K_k = f_k J_k. \tag{4.12}$$

One must then either *estimate* for f_k from some ad hoc geometric relationship, determine f_k from an approximate flux-limited theory [9], or *compute* f_k, which requires a solution of the full angle- (and frequency-) dependent transfer equation (4.9). For strict consistency to $O(v/c)$, only the latter course is open because an attempt to specify f_k a priori is likely to introduce errors as large as (or larger than) the other terms we wish to retain,

The standard approach to solving Eq. (4.9) centers on writing stable-difference formulas in space and time. A variety of schemes have been developed; see, e.g., Richtmyer and Morton [18, Chap. 9]. A difficulty with these schemes is that they often yield inaccurate values for the flux because one must compute $I(+\mu)$ and $I(-\mu)$ separately and then subtract in order to find H. In what follows we shall use an alternative approach widely used in astrophysics that recasts the transfer equation into a form that closely parallels the moment equations.

We can obtain a stable and economical solution of (4.9) by transformation to Feautrier variables [4]. Define the mean intensity-like variable

$$j_k(\mu) \equiv \tfrac{1}{2}[I_k(+\mu) + I_k(-\mu)] \qquad (0 \leqslant \mu \leqslant 1) \tag{4.13}$$

and the flux-like variable

$$h_k(\mu) \equiv \tfrac{1}{2}[I_k(+\mu) - I_k(-\mu)] \qquad (0 \leqslant \mu \leqslant 1), \tag{4.14}$$

then taking the symmetric and antisymmetric average of (4.9) for $\pm\mu$ we obtain

$$\frac{1}{c}\frac{\partial j_k}{\partial t} + \mu\frac{\partial h_k}{\partial z} = \kappa_k(B_k - j_k) + \mu\beta\kappa_k h_k, \tag{4.15}$$

and

$$\frac{1}{c}\frac{\partial h_k}{\partial t} + \mu\frac{\partial j_k}{\partial z} = -\kappa_k h_k + \mu\beta\kappa_k(j_k + 3\tilde{B}_k). \tag{4.16}$$

This system of equations has several interesting properties. First, it strongly resembles the moment equations (4.10) and (4.11), but unlike those equations this pair of equations closes, i.e., contains only $j(\mu)$ and $h(\mu)$. Of course, it is angle dependent, and must be solved for various values of μ. Second, the moments J, H, and K are directly calculable from j and h as

$$J_k = \int_0^1 j_k(\mu)\, d\mu, \tag{4.17}$$

$$H_k = \int_0^1 h_k(\mu)\mu\, d\mu, \tag{4.18}$$

and

$$K_k = \int_0^1 j_k(\mu)\, \mu^2\, d\mu. \tag{4.19}$$

Because h_k appears explicitly in the equations, it can be calculated accurately. Hence, it is possible to obtain accurate values of the flux H_k without severe numerical cancellation, as would occur if one solves Eq. (4.9) directly for $I_k(+\mu)$ and $I_k(-\mu)$ and then subtracts before integrating over μ. Third, as we shall show below, for $\beta = 0$ these equations yield both the wave equation and the time-dependent diffusion equation in the optically thin and thick limits, respectively.

As we shall show in § V, one can immediately write stable-difference approximations for (4.15) and (4.16); however, these in general, have only first-order accuracy in the space derivatives when nonuniform stepsizes are used, and/or the opacity varies as a function of depth. In many applications it is important to achieve second-order accuracy in the space derivatives, particularly in static media, or where there are large scattering terms (or departures from LTE). To see why this would be so, we notice that in a static medium $(dH_\nu/d\tau_\nu) = J_\nu - S_\nu$, where $d\tau_\nu = -\kappa_\nu dz$ and S_ν is the source function. Suppose S_ν has a scattering term and is of the general form $S_\nu = (1-\varepsilon) J_\nu + \varepsilon B_\nu$. Then

$$d^2K_\nu/d\tau_\nu^2 = \varepsilon(J_\nu - B_\nu). \tag{4.20}$$

At great depth in the atmosphere $K_\nu \to \frac{1}{3}J_\nu$, hence, (4.20) implies that

$$J_\nu = B_\nu + (\tfrac{1}{3}\varepsilon)(d^2J_\nu/d\tau_\nu^2). \tag{4.21}$$

From (4.20) or (4.21), we see that the difference $(J_\nu - B_\nu)$, which enters directly into the equation of energy balance [cf. Eq. (3.10)], i.e., the equation of radiative equilibrium for a static medium depends explicitly upon the second derivative of J_ν. Errors in this derivative lead directly to errors in energy balance. Moreover, if we attempt to solve (4.21) for J_ν as a differential equation, then errors in $\frac{1}{3}(d^2J_\nu/d\tau_\nu^2)$ compete directly with the thermal source term εB, which can be quite small when ε is small; such errors are thus equivalent to spurious source terms, and can completely falsify the solution for J_ν. Indeed, it is failure to obtain an accurate representation of the second-derivative terms that lies at the root of most difficulties encountered in numerical attempts at solving the transfer equation itself (i.e., *not* analytically reduced to a diffusion equation) in the diffusion regime.

To obtain a second-order scheme we proceed as in the method of lines, and immediately discretize the time derivatives while leaving the space derivatives in continuous form. As we shall find in § V, a backwards time difference is always unconditionally stable, so we center both (4.15) and (4.16) at t^n and $(1/c)(\partial h_k/\partial t) = (h_k^n - h_k^{n-1})/c\Delta t$ and similarly for $(\partial j_k/\partial t)$.

For economy of notation we define $\gamma \equiv (1/c\Delta t)$, and drop the superscript n, writing $h_k^n \equiv h_k$ and $h_k^{n-1} \equiv h_k^-$. We then have

$$\mu(\partial h_k/\partial z) = \kappa_k B_k - (\kappa_k + \gamma) j_k + \mu\beta\kappa_k h_k + \gamma j_k^- \tag{4.22}$$

and

$$\mu(\partial j_k/\partial z) = -(\kappa_k + \gamma)\, h_k + \mu\beta\kappa_k(j_k + 3\tilde{B}_k) + \gamma h_k^-. \tag{4.23}$$

Now defining $d\tau_k \equiv -(\kappa_k + \gamma)\, dz$ and $\alpha_k \equiv \kappa_k/(\kappa_k + \gamma)$, we obtain

$$\mu(\partial h_k \partial\tau_k) = j_k - \alpha_k B_k - \mu\beta\alpha_k h_k - (1 - \alpha_k)\, j_k^- \tag{4.24}$$

and

$$\mu(\partial j_k/\partial\tau_k) = h_k - \mu\beta\alpha_k(j_k + 3\tilde{B}_k) - (1 - \alpha_k)\, h_k^-. \tag{4.25}$$

Solving (4.25) for h_k and substituting into (4.24) we obtain the second-order system

$$\begin{aligned} \mu^2 \frac{\partial^2 j_k}{\partial\tau_k^2} &+ 2\mu^2\beta\alpha_k \frac{\partial j_k}{\partial\tau_k} \\ &= \left[1 - \mu^2 \frac{\partial(\beta\alpha_k)}{\partial\tau_k}\right] j_k - \alpha_k B_k - (1 - \alpha_k)(j_k^- + \mu\beta\alpha_k h_k^-) \\ &\quad - \mu \frac{\partial}{\partial\tau_k} [3\mu\beta\alpha_k\tilde{B}_k + (1 - \alpha_k)\, h_k^-] + O(\beta^2) \end{aligned} \tag{4.26}$$

which is of the general form

$$\mu^2 \frac{\partial^2 j_k}{\partial\tau_k^2} + a_k \frac{\partial j_k}{\partial\tau_k} = b_k j_k - S_k. \tag{4.27}$$

Because $\beta \ll 1$, $\alpha_k \approx 1$, hence $a_k \ll 1$, $b_k \approx 1$, and $S_k \approx B_k$.

In the limit of a static medium ($\beta \equiv 0$), Eq. (4.27) reduces to the standard second-order form used extensively in stellar atmospheres work (cf., [12, Chaps. 6, 7, 12]), where it is known to be accurate and highly stable in both the free-wave and diffusion regimes. We discuss difference approximations to (4.26) in § V.

Both the first-order system (4.15) and (4.16) and the second-order equation (4.27) require boundary conditions. Physically this means we must assume that $I_- \equiv I(-\mu)$

at the upper boundary and $I_+ \equiv I(+\mu)$ at the lower boundary are given. In some cases one can the diffusion approximation at the lower boundary and write $I_+ = B - (\mu/\kappa)(\partial B/\partial z)$. The mathematical expression of the boundary conditions depends upon the form of the differencing scheme used and will be discussed in § V.

D. *The Free-Wave Limit*

The free-wave limit is obtained when $\kappa \equiv 0$. Equations (4.15) and (4.16) then yield

$$\frac{1}{c^2}\frac{\partial^2 j_k}{\partial t^2} - \mu^2 \frac{\partial^2 j_k}{\partial z^2} = 0 \tag{4.28}$$

and

$$\frac{1}{c^2}\frac{\partial^2 h_k}{\partial t^2} - \mu^2 \frac{\partial^2 h_k}{\partial z^2} = 0 \tag{4.29}$$

which are the standard wave equations. The solutions are of the form

$$j_k \text{ (or } h_k) = Af_1(z + \mu ct) + Bf_2(z - \mu ct), \tag{4.30}$$

i.e., the wave propogates with velocity c along a ray whose angle-cosine is $\pm\mu$ with respect to the normal.

E. *The Diffusion Limit*

In the diffusion limit the term $(1/c)(\partial h/\partial t)$ in Eq. (4.16) is $O(v^2/c^2)$ relative to $\mu(\partial j/\partial z)$. Furthermore, because we expect $\beta \ll 1$ at great depth we drop the β-dependent terms and substitute $h = -(\mu/\kappa)(\partial j/\partial z)$ into (4.15), which yields

$$\frac{1}{c}\frac{\partial j_k}{\partial t} = \frac{\partial}{\partial z}\left(\frac{1}{\kappa_k}\frac{\partial j_k}{\partial z}\right) + \kappa_k(B_k - j_k) + O(\beta). \tag{4.31}$$

Equation (4.31) is of the general form $(\partial f/\partial t) = \nabla^2 f + s$ and, hence, is essentially a time-dependent diffusion equation with a source term.

V. Method of Solution

We now examine various differencing schemes for solving Eqs. (4.15) and (4.16) or Eq. (4.27). In all cases we assume for the present that material properties and the velocity-field are *given*, so that we are performing only a *formal solution* for all angles and frequencies. A sketch of how the transfer problem can be coupled to hydrodynamics will be given in § VI.

A. *First-Order Schemes*

Consider first the first-order equations (4.15) and (4.16). Discretize the problem by dividing the material into D slabs with boundaries located at $\{Z_d\}$, $d = 1,..., D+1$, with $Z_d > Z_{d+1}$, i.e., $d = 1$ is the "top" of the medium and $d = D+1$ is the "bottom." Further, introduce a discrete set of angles $\{\mu_m\}$, chosen according to some quadrature scheme. Frequencies have already been discretized into groups by the mesh $\{\nu_k\}$. Material properties and the mean intensity-like variable $j(\nu)$ are specified at cell centers $Z_{d+1/2}$, $d = 1,..., D$. Velocities and the flux-like variable $h(\nu)$ are specified at cell boundaries Z_d. With this choice of centering the radiation-energy density and pressure are located at the same position as the material-energy density and pressure, while the radiative flux, and hence, the radiation force, will be computed at cell boundaries, where the fluid accelerations are to be calculated. For economy of notation we shall reference only depth levels and time levels t^n, and suppress reference to angles μ_m and frequencies ν_k in assigning subscripts and superscripts to variables.

1. *An Explicit Scheme*

Centering the variables $j^n_{d+1/2}$ at times t^n and the variables $h^{n+1/2}_d$ at times $t^{n+1/2} \equiv \frac{1}{2}(t^n + t^{n+1})$ we can write a standard leapfrog representation of (4.15) and (4.16) as

$$j^{n+1}_{d+1/2} - j^n_{d+1/2} \quad + \mu \quad h^{n+1/2}_d - h^{n+1/2}_{d+1}$$

[illegible]

$$= \kappa^{n+1/2}_{d+1/2}[B^{n+1/2}_{d+1/2} - \tfrac{1}{2}(j^n_{d+1/2} + j^{n+1}_{d+1/2})]$$

$$+ \tfrac{1}{2}\mu\kappa^{n+1/2}_{d+1/2}(\beta^{n+1/2}_d h^{n+1/2}_d + \beta^{n+1/2}_{d+1} h^{n+1/2}_{d+1}) \qquad (d = 1,..., D), \qquad (5.1)$$

and

$$\frac{h^{n+1/2}_d - h^{n-1/2}_d}{c\,\Delta t^n} + \mu \frac{j^n_{d-1/2} - j^n_{d+1/2}}{\Delta Z_{d-1/2}}$$

$$= -\tfrac{1}{2}\kappa^n_d(h^{n-1/2}_d + h^{n+1/2}_d) + \mu\beta^n_d[3(\kappa B)^n_d + (\kappa j)^n_d] \qquad (d = 2,..., D). \quad (5.2)$$

Here $\Delta t^n \equiv \frac{1}{2}(\Delta t^{n-1/2} + \Delta t^{n+1/2})$, $\kappa^{n+1/2}_{d+1/2} \equiv \frac{1}{2}(\kappa^n_{d+1/2} + \kappa^{n+1}_{d+1/2})$, $B^{n+1/2}_{d+1/2} \equiv \frac{1}{2}(B^n_{d+1/2} + B^{n+1}_{d+1/2})$, and $\beta^n_d \equiv \frac{1}{2}(\beta^{n-1/2}_d + \beta^{n+1/2}_d)$. The term ΔZ_d is the thickness of the dth slab and $\Delta Z_{d-1/2} \equiv \frac{1}{2}(\Delta Z_{d-1} + \Delta Z_d)$. Further, $\kappa^n_d \equiv (\Delta Z_{d-1}\kappa^n_{d-1/2} + \Delta Z_d \kappa^n_{d+1/2})/(\Delta Z_{d-1} + \Delta Z_d)$, and similarly for $(\kappa B)^n_d$ and $(\kappa j)^n_d$.

To apply the boundary conditions we use Eq. (5.2) over the half intervals from cell edge to center for the first and last slab and make use of the identities $j^n_1 \equiv I^n_- + h^n_1 = I^n_- + \frac{1}{2}(h^{n-1/2}_1 + h^{n+1/2}_1)$ and $j^n_{D+1} \equiv I^n_+ - h^n_{D+1} = I^n_+ - \frac{1}{2}(h^{n-1/2}_{D+1} + h^{n+1/2}_{D+1})$. We then have

$$\frac{h_1^{n+1/2} - h_1^{n-1/2}}{c\,\Delta t^n} + \mu \frac{I_-^n + ((h_1^{n-1/2} + h_1^{n+1/2})/2) - j_{3/2}^n}{\Delta Z_1/2} = -\kappa_{3/2}^n (h_1^{n-1/2} + h_1^{n+1/2})/2$$
$$+ \mu\beta_1^n \kappa_{3/2}^n [3B_{3/2}^n + I_-^n + (h_1^{n-1/2} + h_1^{n+1/2})/2], \tag{5.3}$$

and

$$\frac{h_{D+1}^{n+1/2} - h_{D+1}^{n-1/2}}{c\,\Delta t^n} + \mu \frac{j_{D+1/2}^n - I_+^n + (h_{D+1}^{n-1/2} + h_{D+1}^{n+1})/2}{\Delta Z_D/2} = -\kappa_{D+1/2}^n (h_{D+1}^{n-1/2} + h_{D+1}^{n+1/2})/2$$
$$+ \mu\beta_{D+1}^n \kappa_{D+1/2}^n [3B_{D+1/2}^n + I_+^n - (h_{D+1}^{n-1/2} + h_{D+1}^{n+1/2})^2]. \tag{5.4}$$

Equation (5.1) provides D equations for updating the D values of $j_{d+1/2}^{n+1}$, and Eqs. (5.2)–(5.4) provide $D+1$ equations for updating the $D+1$ values of $h_d^{n+1/2}$. It should be noted that in the updating procedure one can vectorize the solution either over the depth-grid, or over all angles and frequencies.

Having calculated the solution for all angles, the moments for frequency group k are

$$(E_R)_{d+1/2,k}^{n+1} = \frac{4\pi}{c} J_{d+1/2,k}^{n+1} = \frac{4\pi}{c} \sum_m a_m j_{d+1/2,m,k}^{n+1}, \tag{5.5}$$

$$F_{d,k}^{n+1/2} = 4\pi H_{d,k}^{n+1/2} = 4\pi \sum_m a_m \mu_m h_{d,m,k}^{n+1/2}, \tag{5.6}$$

and

$$(p_R)_{d+1/2,k}^{n+1} = \frac{4\pi}{c} K_{d+1/2,k}^{n+1} = \frac{4\pi}{c} \sum_m a_m \mu_m^2 j_{d+1/2,m,k}^{n+1}, \tag{5.7}$$

where the a_m are appropriate angle-quadrature weights.

The von Neumann local stability analysis shows that (5.1) and (5.2) are stable for

$$c\,\Delta t < \Delta Z/\mu, \tag{5.8}$$

i.e., the usual Courant condition. Although this scheme is easy and cheap for *radiation-flow* problems where the relevant time steps are indeed those given by (5.8), it is clearly less useful for *fluid-flow* problems where one would prefer time steps of the order of $\Delta t \sim \Delta Z/v$, where $v \ll c$ is the fluid velocity. Further, even though the stability analysis does not indicate a restriction on the optical-depth step size, the fact that one can recast the system into a diffusion equation [cf., Eq. (4.31)] suggests that in the diffusion limit a more stringent time-step limitation of the form $\Delta t \leqslant k(\Delta Z)^2$ may come into operation. In either case an implicit method is indicated.

2. *Two-Level Implicit Schemes*

To obtain an implicit scheme, we employ the same spatial centering described above, but put all variables at a common time level. We then can represent (4.15) and (4.16) as

$$\frac{j^{n+1}_{d+1/2} - j^{n}_{d+1/2}}{c\,\Delta t^{n+1/2}} + \theta\mu \frac{h^{n+1}_{d} - h^{n+1}_{d+1}}{\Delta Z_d} + (1-\theta)\mu \frac{h^{n}_{d} - h^{n}_{d+1}}{\Delta Z_d}$$
$$= \theta\kappa^{n+1}_{d+1/2}(B^{n+1}_{d+1/2} - j^{n+1}_{d+1/2})$$
$$+ (1-\theta)\,\kappa^{n}_{d+1/2})(B^{n}_{d+1/2} - j^{n}_{d+1/2}) + \tfrac{1}{2}\theta\mu\kappa^{n+1}_{d+1/2}(\beta^{n+1}_{d} h^{n+1}_{d} + \beta^{n+1}_{d+1} h^{n+1}_{d+1})$$
$$+ \tfrac{1}{2}(1-\theta)\,\mu\kappa^{n}_{d+1/2}(\beta^{n}_{d} h^{n}_{d} + \beta^{n}_{d+1} h^{n}_{d+1}) \qquad (d = 1,\ldots, D), \tag{5.9}$$

and

$$\frac{h^{n+1}_{d} - h^{n}_{d}}{c\,\Delta t^{n+1/2}} + \theta\mu \frac{j^{n+1}_{d-1/2} - j^{n+1}_{d+1/2}}{\Delta Z_{d-1/2}} + (1-\theta)\mu \frac{j^{n}_{d-1/2} - j^{n}_{d+1/2}}{\Delta Z_{d-1/2}}$$
$$= -\theta\kappa^{n+1}_{d} h^{n+1}_{d} - (1-\theta)\,\kappa^{n}_{d} h^{n}_{d} + \theta\mu\beta^{n+1}_{d}[3(\kappa B)^{n+1}_{d} + (\kappa j)^{n+1}_{d}]$$
$$+ (1-\theta)\,\mu\beta^{n}_{d}[3(\kappa B)^{n}_{d} + (\kappa j)^{n}_{d}] \qquad (d = 2,\ldots, D). \tag{5.10}$$

Here auxiliary quantities such as κ_d, $(\kappa B)_d$, and $(\kappa j)_d$ are defined as in the explicit scheme. Boundary conditions are obtained by again applying (5.10) over half intervals of the two end cells. One finds

$$\frac{h^{n+1}_{1} - h^{n}_{1}}{c\,\Delta t^{n+1/2}} + \theta\mu \frac{I^{n+1}_{-} + h^{n+1}_{1} - j^{n+1}_{3/2}}{\Delta Z_1/2} + (1-\theta)\mu \frac{I^{n}_{-} + h^{n}_{1} - j^{n}_{3/2}}{\Delta Z_1/2}$$
$$= -\theta\kappa^{n+1}_{3/2} h^{n+1}_{1} - (1-\theta)\,\kappa^{n}_{3/2} h^{n}_{1}$$
$$+ \theta\mu\beta^{n+1}_{1}\kappa^{n+1}_{3/2}(3B^{n+1}_{3/2} + I^{n+1}_{-} + h^{n+1}_{1}) + (1-\theta)\,\mu\beta^{n}_{1}\kappa^{n}_{3/2}(3B^{n}_{3/2} + I^{n}_{-} + h^{n}_{1}), \tag{5.11}$$

and

$$\frac{h^{n+1}_{D+1} - h^{n}_{D+1}}{c\,\Delta t^{n+1/2}} + \theta\mu \frac{j^{n+1}_{D+1/2} - I^{n+1}_{+} + h^{n+1}_{D+1}}{\Delta Z_D/2} + (1-\theta)\mu \frac{j^{n}_{D+1/2} - I^{n}_{+} + h^{n+1}_{D+1}}{\Delta Z_D/2}$$
$$= -\theta\kappa^{n+1}_{D+1/2} h^{n+1}_{D+1} - (1-\theta)\,\kappa^{n}_{D+1/2} h^{n}_{D+1}$$
$$+ \theta\mu\beta^{n+1}_{D+1}\kappa^{n+1}_{D+1/2}(3B^{n+1}_{D+1/2} + I^{n+1}_{+} - h^{n+1}_{D+1})$$
$$+ (1-\theta)\,\mu\beta^{n}_{D+1}\kappa^{n}_{D+1/2}(3B^{n}_{D-1} + I^{n}_{+} - h^{n}_{D+1}). \tag{5.12}$$

For $\theta = 1$, one has a *fully implicit* or *backward Euler* scheme; for $\theta = \frac{1}{2}$ one has a *Crank-Nicholson* scheme. A von Neumann local stability analysis shows that both schemes are unconditionally stable.

If we choose a solution vector

$$\mathbf{X} = (h_1, j_{3/2}, h_2, \ldots, h_D, j_{D+1/2}, h_{D+1}), \tag{5.13}$$

then at each angle μ_m and frequency ν_k the system (5.9)–(5.12) is of the form

$$\mathbf{T}_{mk}\mathbf{X}_{mk} = \mathbf{R}_{mk}, \tag{5.14}$$

where $\mathbf{T}_{mk}$ is a tridiagonal matrix of dimension $(2D+1)$ and $\mathbf{R}_{mk}$ is a vector of length $(2D+1)$. The computational effort to solve the system scales as $c(2D+1)$. Because the solution is recursive it cannot be vectorized over the depth grid. Both the setup and solution, however, can be vectorized over all angles and frequencies, which, in effect, are treated in parallel. In a scalar machine the total computing effort scales as $cMK(2D+1)$, where M is the number of angles and K the number of frequencies. In a vector machine the coefficient MK is reduced by some large factor depending on the length of the vector registers.

We can write Eq. (5.9) in the form

$$D_d h_d^{n+1} + E_d h_{d+1}^{n+1} = F_d j_{d+1/2}^{n+1} + G_d \qquad (d = 1,..., D) \tag{5.15}$$

and Eq. (5.10) in the form

$$A_d j_{d-1/2}^{n+1} + B_d j_{d+1/2}^{n+1} = C_d h_d^{n+1} + L_d \qquad (d = 2,..., D). \tag{5.16}$$

The boundary conditions are also of the form of Eq. (5.16) with $A_1 = 0$ in the upper boundary condition and $B_{d+1} = 0$ in the lower boundary condition. In general, A_d, B_d, C_d, D_d, E_d, and F_d are matrices of oreder $(M \times M)$ if scattering is included. In the current nonscattering problem, these quantities are simply scalars and can be read by inspection from Eqs. (5.9)–(5.12). The quantities G_d and L_D are vectors and contain all known information from the previous time step in t^n.

In practice, we solve (5.15) and (5.16) by the recursion relations

$$h_{d+1}^{n+1} = V_{d+1} j_{d+3/2}^{n+1} + W_{d+1} \tag{5.17}$$

$$j_{d+1/2}^{n+1} = U_{d+1/2} h_{d+1}^{n+1} + T_{d+1/2}, \tag{5.18}$$

where

$$U_{d+1/2} = (F_d - D_d V_d)^{-1} E_d \tag{5.19}$$

$$V_d = (C_d - A_d U_{d-1/2})^{-1} B_d \tag{5.20}$$

$$W_d = (C_d - A_d U_{d-1/2})^{-1} (A_d T_{d-1/2} - L_d) \tag{5.21}$$

$$T_{d+1/2} = (F_d - D_d V_d)^{-1} (D_d W_d - G_d). \tag{5.22}$$

The solution starts at $d = 1$ with $A_1 = 0$, and we then recursively construct all composite quantities V, W, U, T for all depth points until we reach $d = D+1$, where $B_{d+1} = 0$. The bottom boundary condition implies $h_{D+1} = W_{d+1}$. We then back substitute using Eqs. (5.17) and (5.18) to obtain $j_{d+1/2}^{n+1}$ $(d = 1,..., D)$ and h_d^{n+1} $(d = 1,..., D+1)$.

An alternative solution to the coupled system is to develop a tridiagonal system in just one variable. Direct substitution of Eq. (5.16) into Eq. (5.15) yields

$$(D_d C_d^{-1} A_d)\, j_{d-1/2}^{n+1} + (D_d C_d^{-1} B_d + E_d C_d A_d - F_d)\, j_{d+1/2}^{n+1} + (E_d C_d^{-1} B_d)\, j_{d+3/2}^{n+1}$$
$$= (D_d + E_d)\, C_d^{-1} L_d + G_d. \tag{5.23}$$

We solve this tridiagonal system for $j_{d+1/2}^{n+1}$ and use Eq. (5.16) to obtain

$$h_d^{n+1} = C_d^{-1}(A_d j_{d-1/2}^{n+1} + B_d j_{d+1/2}^{n-1} - L_d). \tag{5.24}$$

Having obtained the solution, moments can be constructed as in Eqs. (5.5)–(5.7).

B. *The Second-Order Scheme*

Let us now briefly examine the second-order system (4.26). The primary goal is to obtain a second-order accurate representation of the differential operators on the left-hand side. Let $\Delta\tau_d$ and $\Delta\tau_{d+1}$ denote the τ-increments between $d-\frac{1}{2}$ and $d+\frac{1}{2}$, and $d+\frac{1}{2}$ and $d+\frac{3}{2}$, respectively, and let $\Delta\tau_{d+1/2} \equiv \frac{1}{2}(\Delta\tau_d + \Delta\tau_{d+1})$. Represent the depth-variation of j by a second-order Lagrange-interpolation polynomial. Then second-order formulae for $(\partial^2 j/\partial\tau^2)$ and $(\partial j/\partial\tau)$ evaluated at $d+\frac{1}{2}$ are

$$(\partial^2 j/\partial\tau^2)_{d+1/2} = \frac{j_{d-1/2}}{\Delta\tau_d\,\Delta\tau_{d+1/2}} - \frac{j_{d+1/2}}{\Delta\tau_{d+1/2}}\left(\frac{1}{\Delta\tau_d} + \frac{1}{\Delta\tau_{d+1}}\right)$$
$$+ \frac{j_{d+3/2}}{\Delta\tau_{d+1}\,\Delta\tau_{d+1/2}} \tag{5.25}$$

$$(\partial j/\partial\tau)_{d+1/2} = \frac{\Delta\tau_d j_{d+3/2}}{2\Delta\tau_{d+1/2}\,\Delta\tau_{d+1}} + \left(\frac{1}{\Delta\tau_d} - \frac{1}{\Delta\tau_{d+1}}\right) j_{d+1/2}$$
$$- \frac{\Delta\tau_{d+1}}{2\Delta\tau_{d+1/2}\,\Delta\tau_d}\, j_{d-1/2}. \tag{5.26}$$

The coefficient of $(\partial j/\partial\tau)$ in Eq. (4.26) and the derivatives on the right-hand side should, in principle, also be obtained to second-order accuracy. Depending on the centering of the various variables this may require careful interpolation, using, e.g., splines of other high-order formulae.

The final system is tridiagonal in form for each angle-frequency choice, and both setup and solution can be vectorized over angles and frequencies. Note that there are now only D variables (the run of j_d with depth) in each system. Once the $j_{d+1/2}$'s have been computed, the fluxes h_d follow immediately from (4.25). In computing $(\partial j/\partial\tau)_d$, one might use a parabolic formula using either $(j_{d-3/2}, j_{d-1/2}, j_{d+1/2})$ or $(j_{d-1/2}, j_{d+1/2}, j_{d+3/2})$, depending on centering, or could numerically fit the j's with a cubic spline and then use the spline-derivative formula. Given j and h, the moments are again calculated by quadrature, as in (5.5)–(5.7).

It should be noted tat the second-order equation (4.26) or (4.27) is essentially computed along a *ray*, and therefore can be adepted to two-dimensional calculations as in Mihalas *et al.* [14]. In our opinion this scheme holds the greatest promise for use in multidimensional geometries.

C. *Test Calculations*

We have examined some of the schemes described above for several test problems.

1. *Unattenuated Square Wave*

For this problem we shall take $\kappa = \beta = B = 0$, and $I_+(t) \equiv 0$ at the lower boundary. We choose an incident intensity $I_- = 0$ for $t < 0$, and $I_- = 1$ for $t \geqslant 0$ with initial conditions $j_{d+1/2} = 0$ and $h_d = 0$ at $t = 0$ (except for $h_1 = -\frac{1}{2}$). The exact solution is a square wave propagating downward in the material, with $j = \frac{1}{2}$ for $s \leqslant \mu ct$ and $j_{d+1/2} = 0$ for $s > \mu ct$, where s measures the distance inward from the boundary.

The results from the explicit scheme with $c\Delta t/\Delta z = 1$ are shown in Figs. 1a–c for $\mu = 1$, 0.5, and 0.25. For $\mu = 1$ (Fig. 1a), where the time step matches the space step exactly, we obtain a perfect square wave. For $\mu = 0.5$ and 0.25 (Figs. 16b and c, respectively) where now $\mu c\Delta t$ does not exactly equal the spatial step, we propagate the wave with the correct velocity (+ indicates nominal half-intensity point of the

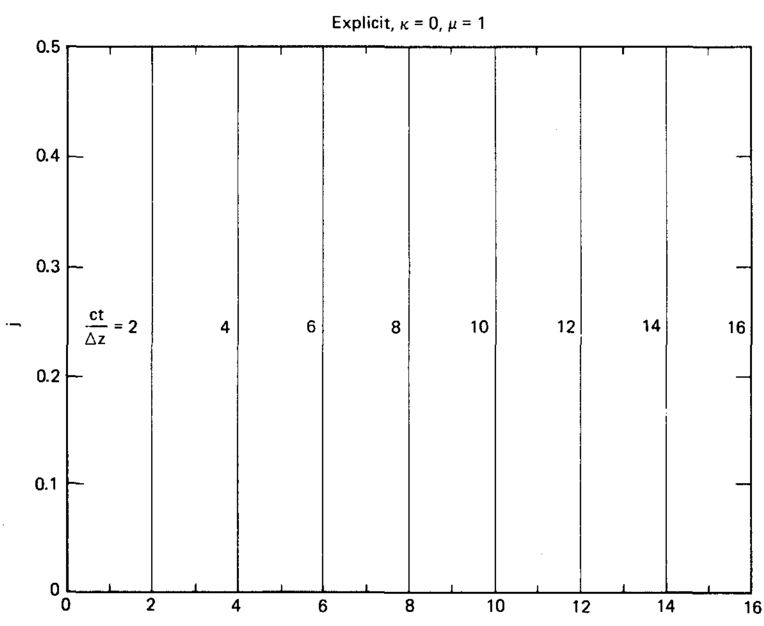
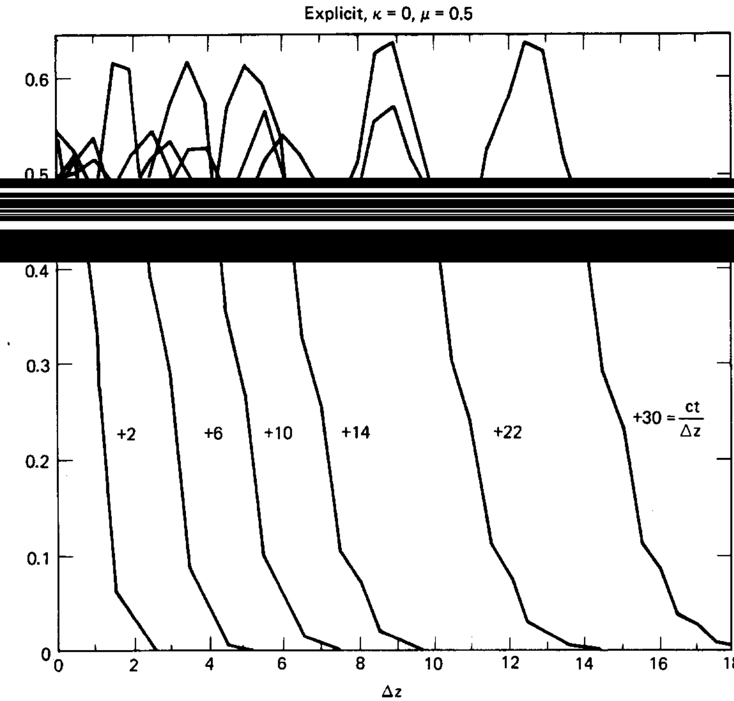


FIG. 1. Propagation of an unattenuated square wave by the explicit scheme using $c\Delta t/\Delta Z = 1$. (a) $\mu = 1$; (b) $\mu = 0.5$; (c) $\mu = 0.25$.



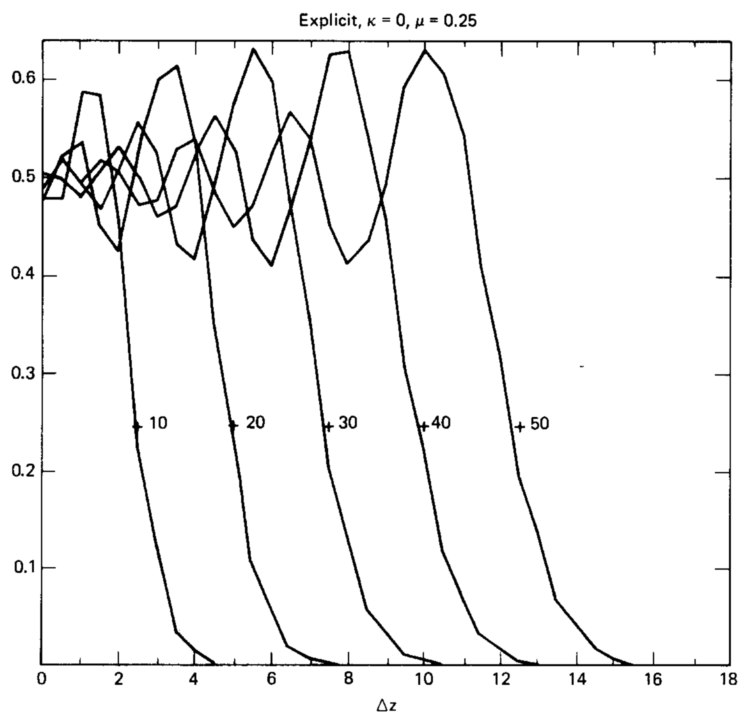


FIGURE 1 *(continued)*

wavefront), but there is a substantial oscillation (±20%) behind the front resulting from an inadequate representation of the higher Fourier components in the square wave.

The solution of this problem given by the fully implicit (backward Euler) and the Cranck–Nicholson schemes are shown in Fig. 2 and (b). The backward Euler scheme (Fig. 1a) gives a completely smooth solution, which propagates at the correct velocity. The wavefront is no longer vertical because this method does not accurately propagate the higher Fourier components in the square wave, and the numerical solution shows both a precursor, and a shoulder behind the wavefront. The Cranck–Nicholson scheme (Fig. 2b) also propagates the wave with nearly the correct velocity (it is systematically slow), and gives a slightly steeper wavefront than the backward Euler scheme. It also has large (±30%) oscillations beind the wavefront however, which would probably do violence to energy balance in dynamical calculations. On the whole the backward Euler scheme is the better of the two.

For $c\Delta t/\Delta z = 10$ the explicit scheme is of course violently unstable, whereas the implicit schemes remain stable. Again the backward Euler scheme (Fig. 3a) propagates a smooth wavefront with about the right speed while the Crank–Nicholson scheme shows severe oscillations (±40%) which are likely unacceptable (Fig. 3b).

2. *Attenuated Radiation Front*

Here we shall take $j = h = 0$ at $t = 0$, $I_+ \equiv 0$, and $I_- = 0$ for $t < 0$ and $I_- = 1$ for $t \geqslant 0$, but now set $\kappa = 0.1$ so that the optical-depth step between mesh points is $\Delta\tau = \kappa\Delta Z = 0.1$. Again, we set $\beta = B = 0$. We thus have a propagating attenuated radiation front. The exact solution is $j = 0.5e^{-\kappa s}$ for $s \leqslant \mu ct$ and $j = 0$ for $s > \mu ct$, where s measures the distance inward from the boundary.

Results from the explicit schem are shown in Figs. 4a–c. For $\mu = 1$ (Fig. 4a) we again obtain a perfectly sharp wavefront, with a very low-amplitude oscillation immediately behind the front; the oscillation rapidly dies out behind the front and the solution approaches the correct limiting result as $t \to \infty$. For $\mu = 0.5$ and 0.25 (Figs.4b and a, respectively) the behavior is qualitatively similar, though the wavefront is no longer sharp and vertical; it still propagates with the correct velocity, however, and approaches the correct result as $t \to \infty$.

Results from the implicit schemes are shows in Figs 5a and b. The front propagates with the correct velocity (of the half-intensity point) in the backward Euler scheme (Fig. 5a), but has a substantial precursor and a shoulder behind the front. In the Cranck–Nicholson scheme (Fig. 5b) there are again unacceptably large oscillations behind the front and a systematic lag of the front behind its nominal position.

3. *Irradiated Grey Atmosphere*

Here we shall take a grey atmosphere in radiative equilibrium, which has an unperturbed mean intensity $J_{\text{grey}}(\tau)$ that is computed by solution of Eqs. (4.15), (4.16) without the time-dependent terms and using a thermal-source term given by

$$B(\tau) = 3H[\tau + q(\tau)], \tag{5.27}$$

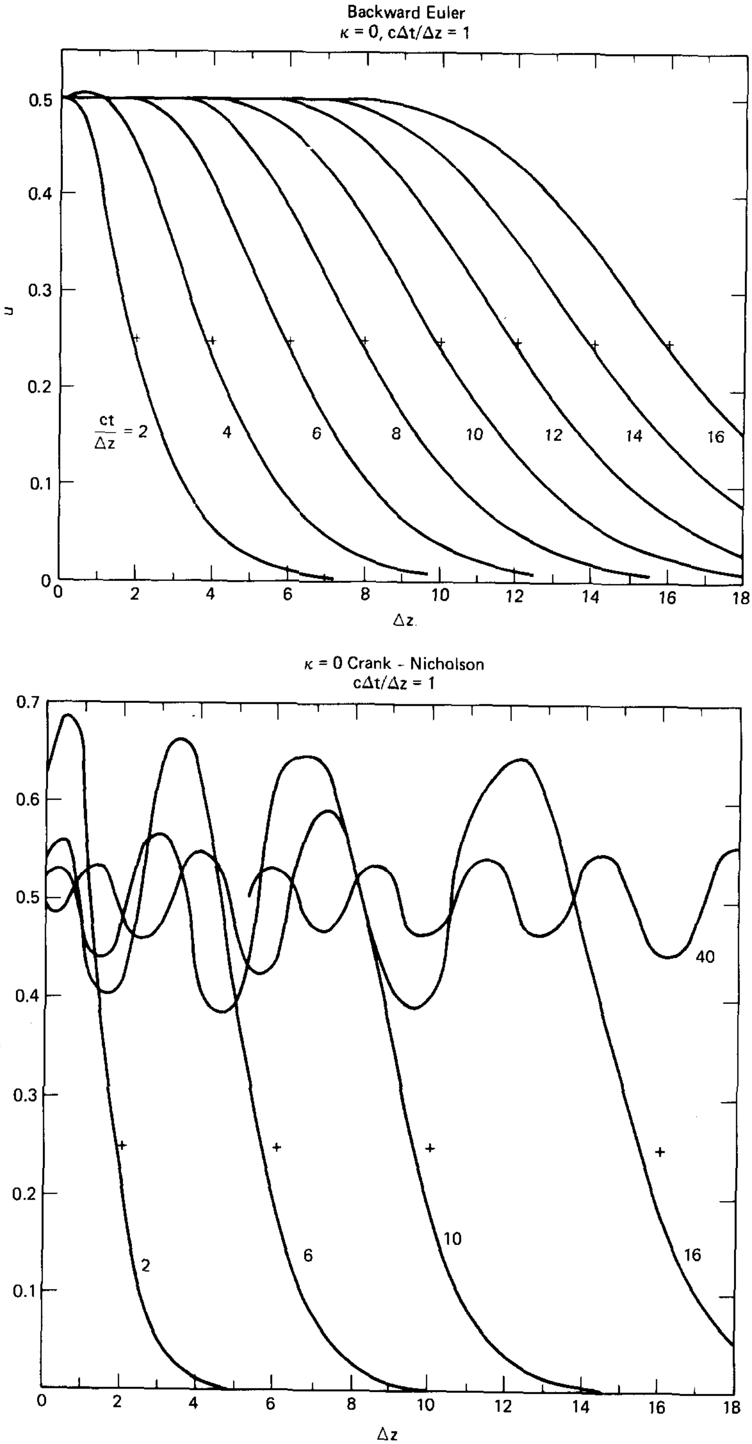


FIG. 2. Propagation of an unattenuated square wave by implicit schemes using $c\Delta t/\Delta Z = 1$. (a) Backward Euler; (b) Crank–Nicholson.

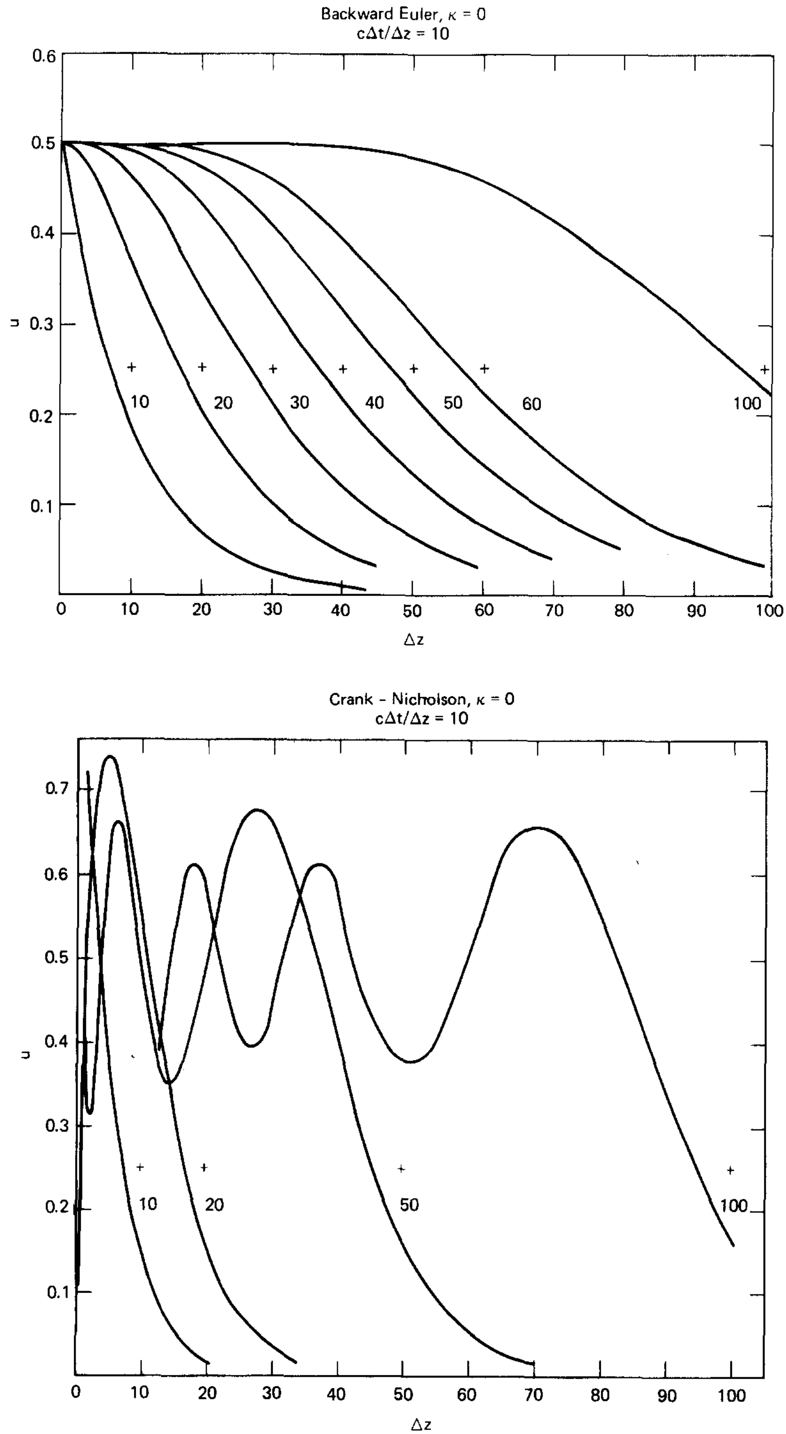


FIG. 3. Propogation of an unattenuated square wave implicit schemes using $c\Delta t/\Delta z = 10$. (a) Backward Euler; (b) Crank–Nicholson.

where $q(\tau)$ is the Hopf function, and a constant flux $H \equiv 1$. For $t \geqslant 0$ we iradiate the atmosphere with an intensity $I_- \equiv I_0$, and follow the time-variation of the internal-radiation field using a three-point double-Gauss angle quadrature. We hold $B(\tau)$ *fixed*, so that the atmosphere is *not* in energy equlibrium.

The incident intensity propagates inward such that for $s \leqslant ct$,

$$I_-(\mu) = I_0 \exp(-\kappa s/\mu) \tag{5.28}$$

for $(s/ct) \leqslant \mu \leqslant 1$, and $I^-(\mu) \equiv 0$ for $s > ct$. This inward intensity gives rise to the following changes in the moments:

$$\Delta J(s) = \tfrac{1}{2} I_0 [E_2(\kappa s) - (s/ct)\, E_2(kct)] \tag{5.29}$$

and

$$\Delta H(s) = -\tfrac{1}{2} I_0 [E_3(\kappa s) - (s/ct)^2 E_3(\kappa ct)] \tag{5.30}$$

for $s \leqslant ct$, and $\Delta J = \Delta H = 0$ for $s > ct$, where $E_n(x)$ is the exponential integral of order n. The complete solution is then $J(s) = J_{\text{grey}} + \Delta J$ and $H(s) = 1 + \Delta H$.

Results from the backward Euler scheme are shown in Figs. 6–8; in all cases we used $c\Delta t/\Delta Z = 1$. We have calculated the exact time-dependent solution using Eqs. (5.29), (5.30) for the case where $I_0 = 1$. These solutions are plotted (---) in Figs. 6a and 7a along with the time-dependent solution of Eqs. (4.15), (4.16) (—). Considering the use of the first-order equations, the accuracy of the solutions is

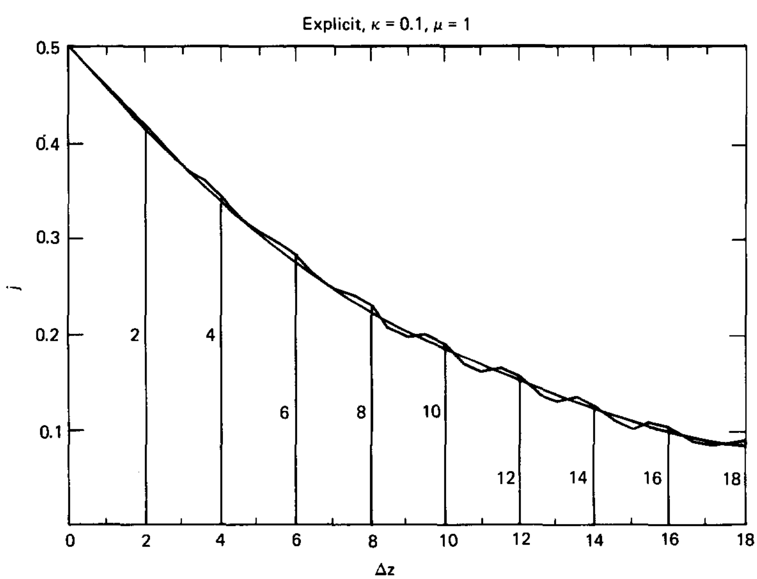
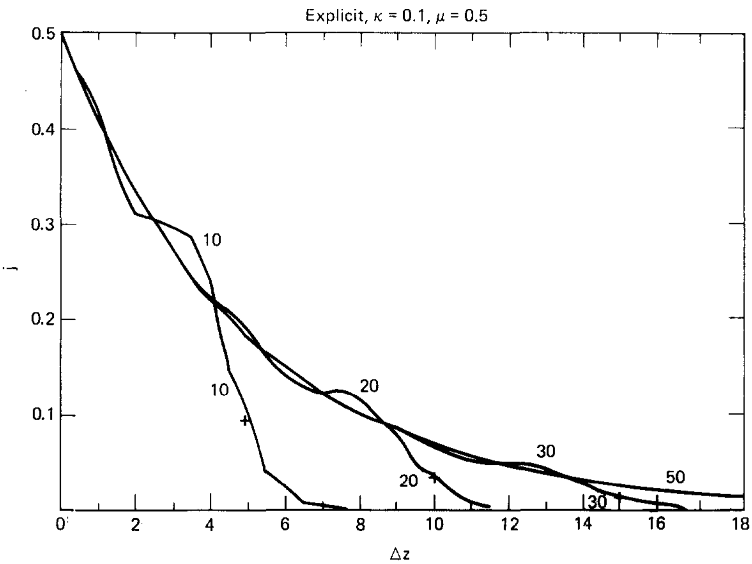


FIG. 4. Propogation of an attenuated radiation front by the explicit scheme using $\kappa = 0.1$, $\kappa\Delta Z = 0.1$, $c\Delta t/\Delta Z = 1$. (a) $\mu = 1$; (b) $\mu = 0.5$; (c) $\mu = 0.25$.



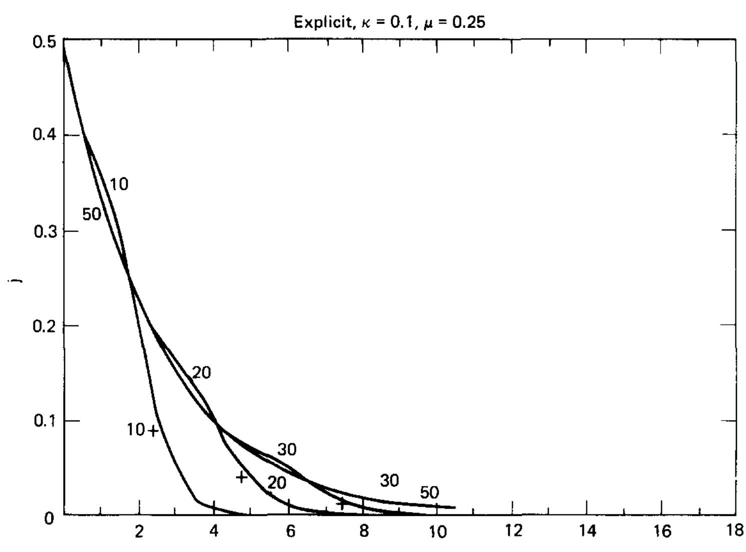


FIGURE 4 *(continued)*

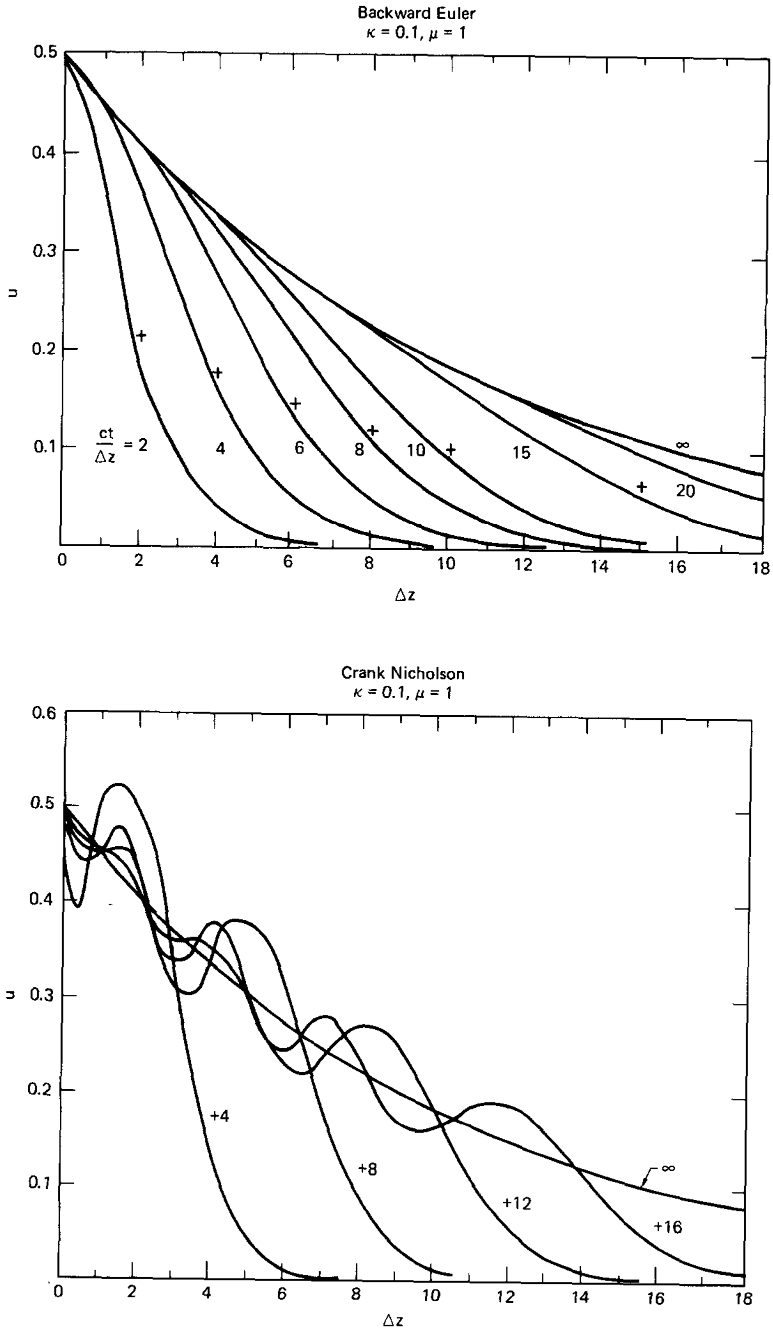


FIG. 5. Propagation of an attenuated radiation front by implicit schemes using $\kappa = 0.1$, $\kappa \Delta Z = 0.1$, $c\Delta t = 1$, $\mu = 1$. (a) Backward Euler; (b) Crank–Nicholson.

remarkably good. The 0th moment J is accurate to better than 1% throughout most of the optical-depth range, while the average error in H is $\approx 2\%$ except for very early times. For both J and H, the computed solutions track the exact solutions quite closely without the significant precursor that was in evidence in the propagation of the unattenuated square wave. In Fig. 7a, the net flux near the boundary decreased for $I_0 = 1$, and becomes negative for $I_0 = 100$ (Fig. 7b), because $I(-\mu) > I(+\mu)$ in that case for $\tau \leqslant 2$. If one were to solve the energy balance equation as well, the atmosphere would finally adjust to a state in which $J(\tau) > J_{\text{grey}}(\tau)$, but $H(\tau)$ would again be identically unity. In Figs. 8a and b, one sees how the intensity for small values of μ approach their final values only at much later times than for $\mu \approx 1$; this is, of course, expected because the radiation can penetrate only into a distance of [redacted] (energy balance being determined by J). The effect is mitigated, however, when the material is actually allowed to heat because energy input is re-radiated in all directions by the heated material.

4. *Velocity Effects in a Moving Atmosphere*

To study the effects of velocity fields we considered a time-independent grey atmosphere with an imposed velocity field. The equations to be solved follow from Eqs. (4.15) and (4.16) and are

$$\mu \frac{\partial h}{\partial \tau} = j - B - \mu\beta h \tag{5.31}$$

and

$$\mu \frac{\partial j}{\partial \tau} = h - \mu\beta(j + 3B). \tag{5.32}$$

To begin we choose $\beta \equiv 0$, and solve the grey-atmosphere problem, using a three-point double-Gauss angle quadrature, to find B. The atmosphere has a logarithmic optical-depth mesh with several (~ 10) points per decade. In practice this is done by solving the second-order system.

$$\mu^2(\partial^2 j/\partial\tau^2) = j - J, \tag{5.33}$$

imposing the condition of a fixed flux ($H \equiv 1$) at the lower boundary and using the Rybicki elimination scheme. Then $B \equiv J$ as obtained from Eq. (5.33). With this value of B we shall perform a formal solution of Eqs. (5.31) and (5.32) to obtain the moments J_0, H_0, and K_0 in the stationary atmosphere. Ideally one has $H_0 = 1$, $J_0 = 3[\tau + q(\tau)]$, and $K_0 = \tau + q(\infty)$.

Knowing B, we can solve Eqs. (5.31) and (5.32) for $\beta \neq 0$, and then determine J,

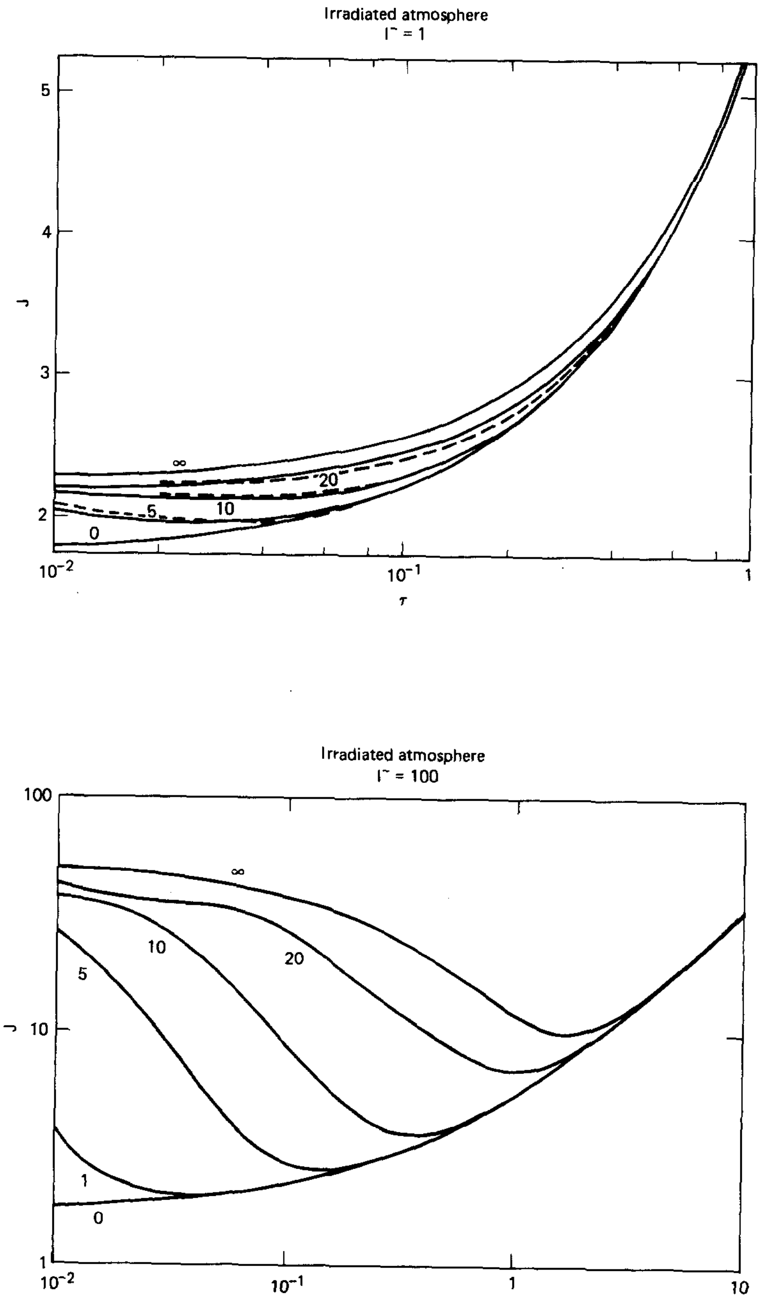


FIG. 6. Mean intensity in an iradiated grey atmosphere. Backward Euler scheme with $c\Delta t/\Delta Z = 1$. (a) $I_0 = 1$; (b) $I_0 = 100$. (---), exact solution.

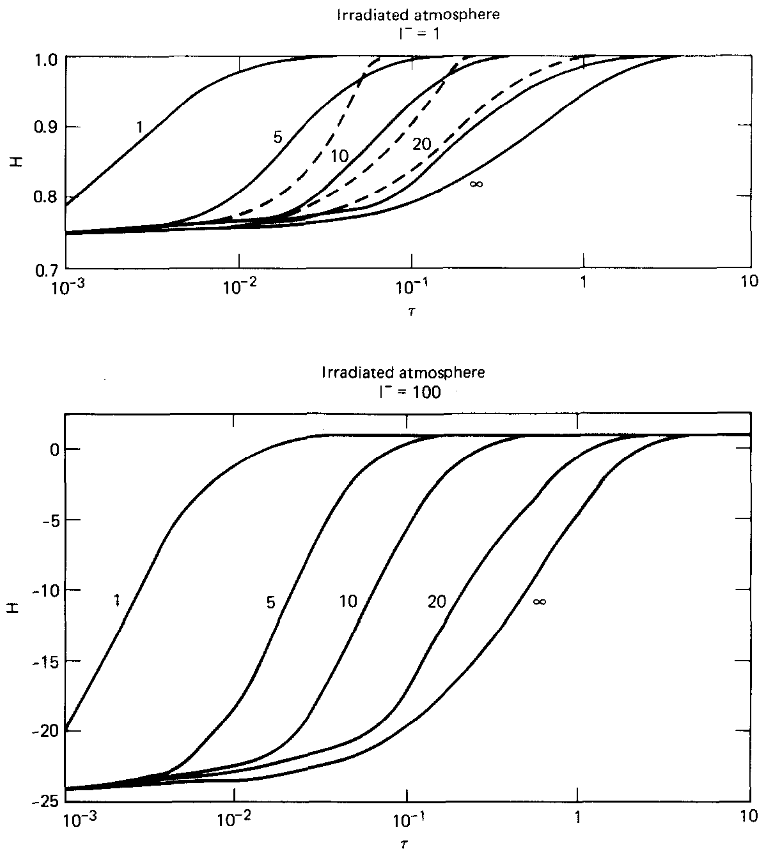


FIG. 7. Flux in an irradiated grey atmosphere. Backward Euler scheme with $c\Delta t/\Delta Z = 1$. (a) $I_0 = 1$; (b) $I_0 = 100$. (---), exact solution.

H, and K from $j(\mu)$ and $h(\mu)$. As a test, we can set $\beta = \text{constant}$, in which case J, H, and K should be related to J_0, H_0, and K_0 by the Lorentz tranformations

$$J_L = J_0 + 2\beta H_0, \tag{5.34}$$

$$H_L = H_0 + \beta(J_0 + K_0), \tag{5.35}$$

and

$$K_L = K_0 + 2\beta H_0. \tag{5.36}$$

Notice that these equations imply a large change between H_0 and for $\tau \gg 1$, because both J_0 and $K_0 \gg H_0$ at depth. For $\tau \ll 1$, J_0, H_0, and K_0 all are of the same order of magnitude, and the velocity-induced changes are thus $O(\beta)$.

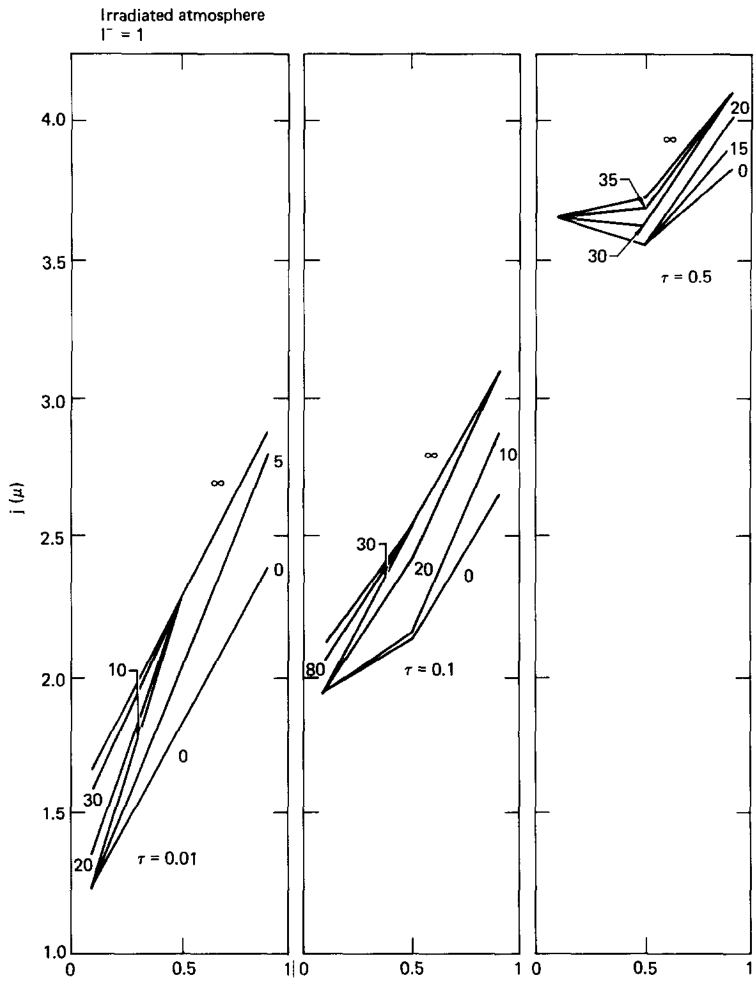


FIG. 8. Angular distribution of specific intensity at selected optical depths in an irradiated grey atmosphere. (a) $I_0 = 1$; (b) $I_0 = 100$.

A quantitative estimate of the importance of velocity-induced effects is given by the ratio

$$\delta_x = (X - X_0)/X_0 \tag{5.37}$$

and of the accuracy of the solution by

$$\varepsilon_x \equiv (X - X_L)/(X - X_0), \tag{5.38}$$

where X stands for J, H, or K.

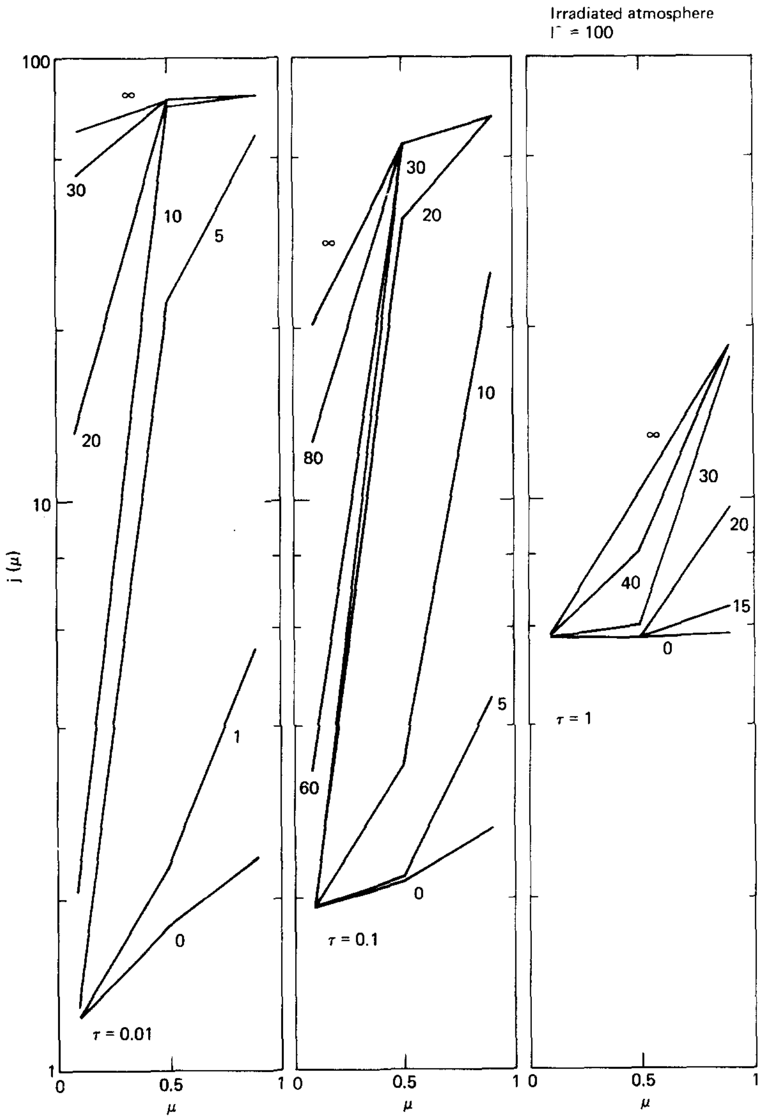


FIGURE 8 *(continued)*

Results for relative change in the flux δH in a uniform flow with $\beta = .01$ are presented in Figs. 9a and b. In Fig. 9a, we see that the inclusion of v/c terms in the transfer equation leads to an increase of 0.03 in H at the surface and remains fairly constant to $\tau \approx 1$. As expected from (5.35), on the range $\tau = 1$, to $\tau = 100$ there is a substantial rise in δ_H caused by the large rise of J and K with depth. Inclusion of v/c terms thus leads to large increases (400%) of the flux compared to the static flux H_0.

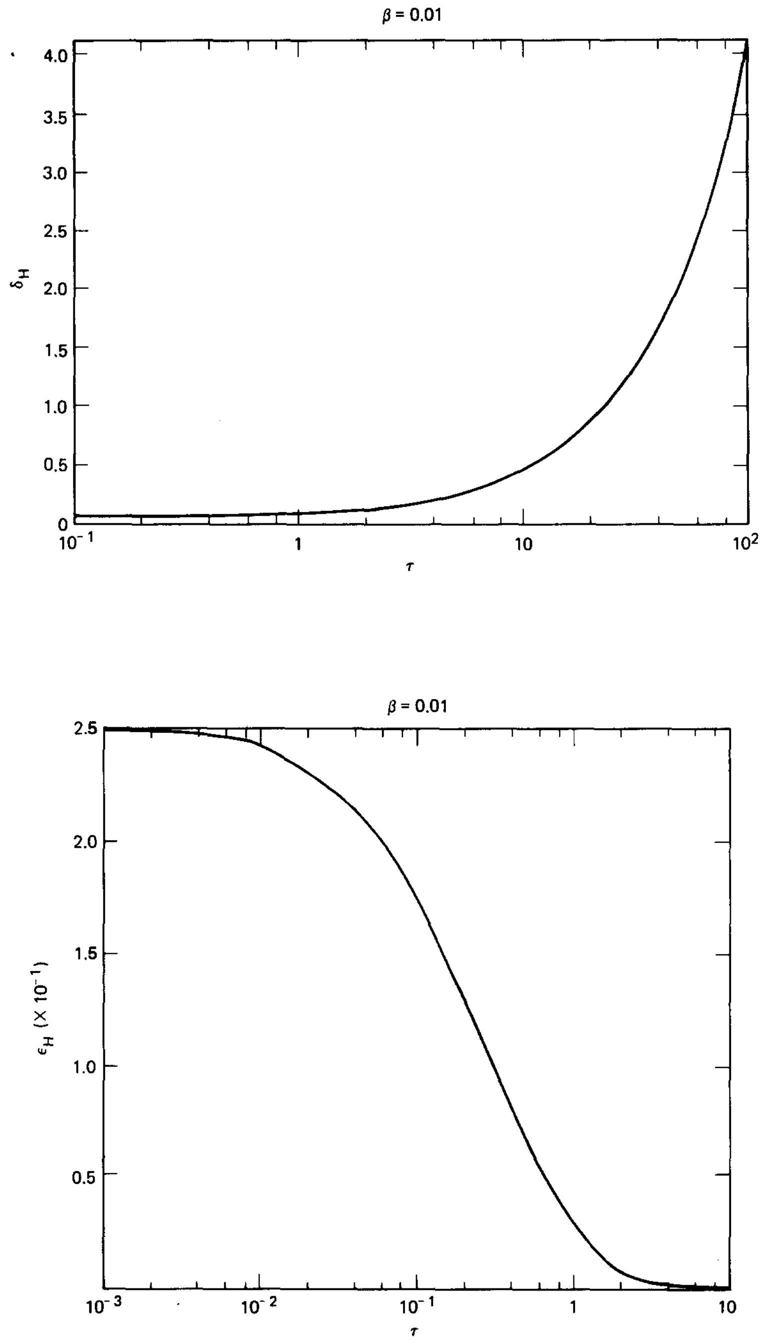


FIG. 9. Velocity dependent solution with $v/c = 0.01$; uniform (a) $\delta_H = (H - H_0)/H_0$ as a function of τ. (b) $\varepsilon_H = (H - H_L)/(H - H_0)$ as a function of τ.

Inspection of the accuracy of our computed solution (Fig. 9b) reveals that we make a 25% error at the surface where the velocity-dependent effects are small (~3%), but only a negligible error ($\ll 1\%$) at depth where the velocity-induced effects are very significant (~400%).

Under the assumption of a uniform flow between the fixed and comoving frames, the calculation of the Lorentz transformation of the integrated moments (Eqs. (5.34)–(5.36)) provides a powerful check on our formulation and solution. Indeed, for the case of constant β the Lorentz-transformed moments in the inertial frame agree to $0(\beta)$ with the computed moments at the surface. At great depth ($\tau \gg 1$), the agreement is $\ll 0(\beta)$. We have verified this by calculation with several values of β. For the case of $\beta = 0$, the solution yields $\delta_H \leqslant 10^{-8}$ as expected. For the variable β case, a preferred frame no longer exists and a simple Lorentz transformation can no longer be used to relate the two frames. Calculations show that the computed moments and the Lorentz transformation do not agree to $O(\beta)$.

VI. Coupling to Hydrodynamics

In this section we shall briefly sketch how the transfer equation including $O(v/c)$ terms might be coupled to the hydrodynamic equations.

A. *Eulerian Formulation*

In a Eulerian formulation we can write the continuity, momentum, and energy equations for one-dimensional flow in conservation-law form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0. \tag{6.1}$$

$$\frac{\partial}{\partial t}\left(\rho v + \frac{4\pi}{c^2}\sum_k w_k H_k\right) + \frac{\partial}{\partial z}\left(\rho v^2 + p + \frac{4\pi}{c}\sum_k w_k f_k J_k\right) = -\rho g + \frac{4\pi\beta}{c}\sum_k w_k \kappa_k (B_k - J_k), \tag{6.2}$$

and

$$\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho v^2 + \frac{4\pi}{c}\sum_k w_k J_k\right) + \frac{\partial}{\partial z}\left[\left(\rho e + \frac{1}{2}\rho v^2 + p\right) v + 4\pi \sum_k w_k H_k\right] = -gv\rho, \tag{6.3}$$

where $f_k \equiv K_k/J_k$ is a variable Eddington factor (presumed known) and k denotes a frequency group. These equations are coupled to the transfer equations

$$\frac{1}{c}\frac{\partial J_k}{\partial t}+\frac{\partial H_k}{\partial z}=\kappa_k(B_k-J_k)+\beta\kappa_k H_k \qquad (k=1,\ldots,K) \tag{6.4}$$

and

$$\frac{1}{c}\frac{\partial H_k}{\partial t}+\frac{\partial(f_kJ_k)}{\partial z}=-\kappa_k H_k+\beta\kappa_k(f_kJ_k+\tilde{B}_k) \qquad (k=1,\ldots,K). \tag{6.5}$$

If we center the variables v and F_κ on slab interfaces and ρ, e, and J_κ at slab centers, Eqs. (6.1) – (6.5) can be discretized into a nonlinear set of algebraic equations. To solve the system we linearize around an approximate solution. The resulting is then block tridiagonal, with blocks of dimensions $(K+3)\times(K+3)$. The computing effort required to solve the system scales as $c(2D)(K+3)^3$. Both the setup and solution is vectorizable. After applying the corrections just obtained to all variables, one can carry out a formal solution for all angles and frequencies by solution of Eqs. (4.15) and (4.16) both to update the radiation field and to obtain new variable Eddington factors. The effort to effect this solution scales as $cDMK$, (M = number of angles). The linearization can then be iterated to convergence.

B. *Lagrangian Formulation*

For one-dimensional flowit is convenient to use Lagrangian variables. Thus writing $dm=-\rho\,dz$ as the mass measured inward into medium, the momentum and energy equations become

$$\frac{Dv}{Dt}=-g+\frac{\partial p}{\partial m}+\frac{4\pi}{c}\sum_k w_k\kappa_k F_k-\frac{4\pi}{c}\beta\sum_k w_k\kappa_k J_k(1+f_k) \tag{6.6}$$

$$\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)=4\pi\sum_k \kappa_k w_k(J_k-B_k)-2\beta(4\pi)\sum_k w_k\kappa_k H_k. \tag{6.7}$$

Provided we can solve the transfer equations in the Lagrangian frame, these equations can be solved by the standard Lagrangian leapfrog scheme, where v is centered at $t^{n+1/2}$, and other variables (including radiation) are centered at t^n, t^{n+1}. Thus if we know p^n, J^n, F^n, etc., we can advance (6.6) from $t^{n-1/2}$ to $t^{n+1/2}$. Using $v^{n+1/2}$ we calculate a new spatial mesh Z^{n+1} at t^{n+1} and, hence, new densities ρ^{n+1}. We then must solve (6.7) implicitly with the transfer equations at t^{n+1}.

To obtain a Lagrangian form of Eq. (4.15) we add and subtract $\beta(j_k/\partial Z)$ from $(1/c)(\partial j_k/\partial t)$, and use the equation of continuity to finally obtain

$$\frac{1}{c}\frac{D}{Dt}(j_k/\rho)-\frac{\partial}{\partial m}(\mu h_k-\beta j_k)=\frac{\kappa_k}{\rho}(B_k-j_k)+\frac{\mu\beta\kappa_k h_k}{\rho}. \tag{6.8}$$

Similarly, for (4.16) we find

$$\frac{1}{c}\frac{D}{Dt}(h_k/\rho)-\frac{\partial}{\partial m}(\mu j_k-\beta h_k)=-\frac{\kappa_k h_k}{\rho}+\frac{\mu\beta\kappa_k}{\rho}(j_k+3\tilde{B}_k). \tag{6.9}$$

The corresponding moment equations are

$$\frac{1}{c}\frac{D}{Dt}\left(\frac{J_k}{\rho}\right)-\frac{\partial}{\partial m}(H_k-\beta J_k)=\frac{\kappa_k}{\rho}(B_k-J_k)+\frac{\beta\kappa_k H_k}{\rho} \tag{6.10}$$

and

$$\frac{1}{c}\frac{D}{Dt}\left(\frac{H_k}{\rho}\right)-\frac{\partial}{\partial m}(f_k J_k-\beta H_k)=-\frac{\kappa_k H_k}{\rho}+\frac{\beta\kappa_k}{\rho}(f_k J_k+\tilde{B}_k). \tag{6.11}$$

Given estimates of the Eddington factors, Eqs. (6.10) and (6.11) are to be solved in parallel with the energy equation (6.7) at the advanced time level t^{n+1}. Again the nonlinear system can be linearized to yield a block-tridiagonal system. After the resulting corrections are applied to the current variables, Eqs. (6.8) and (6.9) are used to carry out a formal solution that gives new Eddington factors. The process can then be iterated to convergence.

The Lagrangian formulation may have important application to the solution of time-dependent neutrino transport during the gravitational collapse of a stellar core resulting in a supernova explosion. To date, the most complete solution to this problem uses a $P-N$ method with flux-limited diffusion in the fluid frame [1]. These solutions are costly to compute and introduce several approximations through use of ad hoc flux limiters as well as leaving out possibly significant v/c terms. In the supernova explosion, $v/c=0.15$ is typical. Our equations become the equations for neutrino transport when the appropriate modifications to the emission and total absorption are made. If conservative scattering is reasonable approximation, our method may provide a more efficient solution with full retention of v/c terms and accurate variable Eddington factors.

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